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STABILITY AND STABILIZATION OF THE WAVE MODEL

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in partial fulfillment of the
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in

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by
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ABSTRACT

The stability properties of 2-D systems are an important aspect of the design of acoustic, seismic, image and sonar signal processors. This research utilizes the Wave model format to transport 1-D stability techniques to the 2-D setting. The research studies stability through multistep growth bounds on the Wave state. The use of Lyapunov theory is also considered.

The research considers also the problem of stabilizing a 2-D system using state and/or output information feedback to interior and/or boundary controls. Finally the problem of observer design for 2-D systems is considered, with the new stability criteria being used to assure observer/system convergence. New results based on symmetrizability are also discussed.

The principal results are illustrated by a number of examples. The results are also interpreted in the context of other contemporary local state models.

CHAPTER 1

INTRODUCTION

In recent years digital signal processing has grown quickly and its application has spread to signals and systems that are characterized by multiple independent variables. In some applications one of the variables (dimensions) could be considered as time while the others represent spatial dimensions. In other applications none of the independent dimensions are necessarily time. Distributive systems and image processing are examples of the above situations respectively.

Among the important multidimensional system (m-D) applications are image processing, geophysics, seismology, X-ray,weather photogrammetry, gravity field analysis, sonar arrays, processing of radar, meteorology, and tomography.

The processing of m-D signals is computationally more complicated than the one dimensional (1-D) case. However, once the transition from one to two dimensions is familiar, the conceptual extension of signal processing methods to m-D is not difficult. On the other hand, there are many properties of 1-D systems which can not be easily generalized to 2-D systems. For this reason 2-D signals and systems remain a challenging and interesting area of research.

The study of 2-D stability is a case in point. The study of 2-D stability is substantially harder than its 1-D counterpart. This is due mainly to the fact that the fundamental theorem of algebra does not extend to 2-D polynomials. This theorem allows one to factor a 1-D characteristic polynomial (of degree n) into a product of n polynomial factors of first degree. The stability of a filter is then found by checking locations of the poles.

In the 2-D case the finite polynomials are not necessarily factorable. Finite polynomials may also have an infinite number of roots. Thus, stability and factorization are distinct questions.

The question of pole placement considers use of compensators to move the system poles to desirable locations. In the 1-D case, pole placement has received ample attention with a resultant comprehensive theory. For instance one powerful theorem states that the poles can be arbitrary placed if and only if the system is completely controllable. The counterpart of this theorem has not been established for 2-D systems. Indeed, as pointed out in [1], the question of controllability, observability, and minimality for 2-D systems are not agreeably posed on a local state basis.

Moreover to find a compensator which results in a specified system characteristic polynomial one must solve an overdefined set of equations. Therefore we can not find the necessary and sufficient conditions for arbitrary pole assignment. However, when the open-loop and closed-loop characteristic polynomial are considered to be separable, we can state necessary and sufficient conditions.

Another issue is minimal realization. In the 1-D case a concise theorem states that a system realization is minimal if and only if it is controllable and observable. Minimal realizations, among other properties, do not have common poles and zeroes. In the 2-D case a concise minimality theorem is not available. Indeed, controllability and observability are not clear matters in the 2-D setting. Secondly, for 2-D polynomials there is a new phenomena, namely the second kind of singularity. This means a common pole-zero point can exist even if the numerator and denominator polynomials are relatively prime.

Several models have been proposed for characterizing m-D maps. All of the models have an obvious iterative characteristic. These models all evidence a quarter plane causal property. These models also have a close resemblence to state models from the 1-D setting.

The first order G-R model was introduced by Givone-Roesser in [2]. In another study Fornasini-Marchesini [3] introduced a second order model motivated by Nerode equivalence. In this model, the global state 'x' arises from the factorization of the 2-D input-output map. The global state has infinite dimension, in general, and preserves all the past information while the local state identifies the recursions to be performed at each step by the 2-D filters. The At model introduced by Attasi in [4] is a special case of the F-M model.

Using the setting of a partially ordered Hilbert Resolution space, Porter-Aravena have developed recursive models for all quarter plane causal linear maps. Among these models the F-T-R model of [5] and the Wave model of [6] are the most relevant to this thesis.

The Wave model has the advantage of being both first order and 1-D. It thus supports studies of minimality, stability and observability. It is also possible to obtain the Wave model equivalent of the F-M, G-R and At models by direct methods. In the present study we in fact focus attention on Wave model equivalents to G-R model.

OUTLINE OF CONTRIBUTIONS

In chapter 4 a new approach to the study of 2-D systems, using the Wave model, is given. This approach enables one to use 1-D techniques for 2-D systems. Multi-step stability is also introduced. Several theorems are stated which express sufficient conditions for stability in terms of the spectral radius of the system matrices. The Lyapunov theory is applied to the Wave model and sufficient conditions for insuring stability are stated. Due to the nature of the Wave model, the Lyapunov equations are time variant, even for shift invariant 2-D systems.

In chapter 5 the stabilization of the Wave model by use of either state feedback or output feedback is considered.

It is not possible to consider pole placement in this case. Indeed the matrix A(n) is not a square matrix and thus the very concept of eigenvalues is not available. However the Wave model provides a useful framework for the feedback stabilization problem. It is also shown that boundary controls do not effect stability of the system.

In chapter 6, the solution of the state observer problem for 2-D system is discussed. Two alternative models, the G-R and Wave are considered. In the G-R format it has been possible to replace the assumption of separability with a nonequivalent condition of symmetry.

The state observer for the Wave model is also considered. With this approach it becomes possible to extend many 1-D results. Furthermore, it is shown that the observer matrix is norm invariant and several theorems related to existence of observer are given. However the stability analysis of the state feedback with the observer suggests a possible interaction which is not present in 1-D systems. Further research is proposed.

CHAPTER 2

MODELS FOR DISCRETE m-D SYSTEMS

2.1 INTRODUCTION

m-D systems are systems that depend on more than one independent variable. One of these variables could be time as in the case of distributed systems or could all be other than time, such as in image processing. Our attention is on 2-D systems in particular.

Several methods are used to represent the operations involved in image processing. Transfer functions, partial difference equations, convolution summations, and state-space model are examples of it. In [7] transfer functions were used for development of linear optics. Habibi [8] described a model for estimating images in presence of noise based on partial difference equations. In [9] a state-space model was introduced for linear iterative circuits. We first discuss the external (input-output) description of a 2-D system. Secondly the state-space models representation will be discussed.

2.2 2-D RECURSIVE FILTERS

2-D digital signal processing is concerned with the processing of discrete signals that are functions of two variables. In order to study 2-D digital filters fruitfully, it is necessary to restrict ourself to the study of certain classes of filters, for instance <u>linear</u> <u>shift-invariant</u> (LSI) digital filters. For (LSI) digital filters the input and output sequences are related by a linear constant difference equation of the form

$$\sum_k \sum_j b_{k,j} y_{m-k,n-j} = \sum_k \sum_j a_{k,j} x_{m-k,n-j} \tag{1}$$

or equivalently, in the frequency domain by the discrete transfer function $H(z_1,z_2)$ defined as

$$H(z_1,z_2) = \frac{Z\{y(m,n)\}}{Z\{x(m,n)\}} = \frac{P(z_1,z_2)}{Q(z_1,z_2)} \tag{2}$$

where $Z\{y(m,n)\}$ and $Z\{x(m,n)\}$ denote the one-sided 2-D Z-transform of the output and the input sequence respectively and are defined to be

$$Z\{y(m,n)\} = Y(z_1,z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} y(m,n) z_1^m \, z_2^n$$

$$Z\{x(m,n)\} = X(z_1,z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x(m,n) z_1^m \, z_2^n \tag{3}$$

Note that in the 1-D case the power of z is negative and some authors are using the equivalent form z_1^{-m} z_2^{-n} for 2-D case.

The inverse Z-transform of $H(z_1,z_2)$ is a 2-D sequence $\{h(m,n)\}$, known as the impulse response of the 2-D filter. If the impulse response has only a finite number of non-zero samples, the corresponding filter is called a finite impulse response (FIR), 2-D filter. Otherwise, it is known as a infinite impulse response (IIR), 2-D filter. For an IIR filter the convolution sum is

$$y(m,n) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x(i,j)h(m-i,n-j). \tag{4}$$

By taking the Z-transform of equation 4 one obtains

$$Y(z_1,z_2) = H(z_1,z_2)X(z_1,z_2)$$

where $H(z_1,z_2)$ is defined in equation 2.

Related to 2-D systems, the following definitions are pertinent.

Def (1): A zero of $Q(z_1,z_2)$ which is not simultaneously a zero of $P(z_1,z_2)$ is called a nonessential singularity of the first kind.

Note that nonessential singularities of the first kind are analogous to poles in 1-D case.

Def (2): A zero of $Q(z_1,z_2)$ which is a zero of $P(z_1,z_2)$ is called a nonessential singularity of the second kind.

Def (3): A 2-D system is said to be causal if when two inputs $x_1(m,n)$ and $x_2(m,n)$ are equal for $m < m_1$, $n < n_1$, then the corresponding outputs $y_1(m,n)$ and $y_2(m,n)$ are equal for $m < m_1$, $n < n_1$.

It is also known that a linear shift-invariant system is causal if and only if the unit-sample response is zero for $(m < 0, n < 0)$.

Def (4): A separable system is a LSI system whose impulse response is a separable sequence.

2.3 THE 2-D STATE SPACE MODELS

In recent years substantial attention has been given to the definition and analysis of stationary 2-D state space models. All of the models are quarter plane causal.

However, nonstationary 2-D systems are untouched. In the following sections we shall summarize a selection of the models that are used most widely in literature. We also discuss how these models are related to each other. Our attention is restricted to the m=2 case and to nonstationary models.

2.3.1 G-R Model

In 1972 Givone-Roesser [2] considered first quadrant-causal systems

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1(i,j) \\ B_2(i,j) \end{bmatrix} u(i,\ j)$$

$$y(i,\ j) = [c_1(i,j) \quad c_2(i,j)] \begin{bmatrix} x^h(i,\ j) \\ x^v(i,\ j) \end{bmatrix} \tag{5}$$

where $x^h \varepsilon R^n$, $x^v \varepsilon R^m$ represent the horizontal and vertical states respectively, u is the input and y the output. The system matrix A is given by

$$A(i,j) = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix}$$

with the submatrices $A_i(i,j)$ $i=1,\dots,4$ of appropriate dimensions.

It is apparent that the map u to y, computed by equation 5, is quarter plane causal.

Considering the stationary case, the 2-D z-transform of equation 5 gives

$$\frac{Y(z_1,z_2)}{U(z_1,z_2)} = C \begin{bmatrix} z_1 I_n - A_1 & -A_2 \\ -A_3 & z_2 I_m - A_4 \end{bmatrix} B$$

$$= \frac{P(z_1,z_2)}{Q(z_1,z_2)}$$

where

$$P(z_1,z_2) = C \ \ \mathrm{adj}\begin{bmatrix} z_1 I_n - A_1 & -A_2 \\ -A_3 & z_2 I_m - A_4 \end{bmatrix} B$$

$$Q(z_1,z_2) = \det \begin{bmatrix} z_1 I_n - A_1 & -A_2 \\ -A_3 & z_2 I_m - A_4 \end{bmatrix}. \qquad (6)$$

2.3.2 F-M Model

Fornasini-Marchesini [3] have considered the 2-D state space model in some detail. The F-M model can be described as follows:

$$x(i+1,\ j+1) = A_1(i,\ j+1)x(i,\ j+1) + A_2(i+1,\ j)x(i+1,\ j) + A_0(i,\ j)x(i,\ j) + B(i,\ j)u(i,\ j)$$

$$y(i,\ j) = c(i,\ j)x(i,\ j) \qquad (7)$$

where $x \varepsilon R^n$, $u \varepsilon R^p$, $y \varepsilon R^q$. Because of the appearance of both $x(i+1,\ j+1)$ and $x(i,\ j)$ terms it is clear that the above equation is not a first-order difference equation. It is thus intuitive that $x(i,j)$ is only a partial state.

2.3.3 <u>Attasi Model</u>

Attasi [4] has considered the following model:

$$x(i+1, j+1) = A_1(i, j+1)x(i, j+1) + A_2(i+1, j)x(i+1, j) - A_1A_2(i, j)x(i, j) + B(i, j)u(i, j)$$

$$y(i, j) = C(i,j)x(i, j) \tag{8}$$

This model is a F-M model with restriction $A_0(i,j) = -A_1A_2(i,j) = -A_2A_1(i,j)$.

We can recast The F-M model as a G-R model by the following manipulations. Defining

$$w(i, j) = x(i, j+1) - A_2(i,j)x(i, j)$$

we then have

$$w(i+1, j) = x(i+1, j+1) - A_2(i,j)x(i+1, j). \tag{9}$$

By substituting equation 7 in 9 we get

$$w(i+1,j) = A_1(i,j+1)x(i,j+1) + A_0(i,j)x(i,j) + B(i,j)u(i,j)$$

therefore we have

$$\begin{bmatrix} w(i+1,j) \\ x(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1(i,j) & [A_0+A_1A_2](i,j) \\ I_n & A_2(i,j) \end{bmatrix} \begin{bmatrix} w(i, j) \\ x(i, j) \end{bmatrix} + \begin{bmatrix} B(i,j) \\ 0 \end{bmatrix} u(i, j)$$

$$y(i, j) = \begin{bmatrix} 0 & C(i,j) \end{bmatrix} \begin{bmatrix} w(i, j) \\ x(i, j) \end{bmatrix}.$$

Thus the Attasi and F-M models can be transformed to the G-R model format.

2.3.4 <u>Modified F-M Model</u>

In a series of studies Porter-Aravena [6] have used the following state model for presenting a given 2-D system

$$x(i+1,j+1) = J(i,j+1)x(i,j+1) + K(i+1,j)x(i+1,j)$$
$$+E(i,j+1)u(i,j+1) + F(i+1,j)u(i+1,j) \qquad (10)$$

where the matrices $J(i,j+1)$, $K(i+1,j)$, $E(i,j+1)$, $F(i+1,j)$ are of appropriate dimensions. The above model reminds of the F-M model which had a term dependent on $x(i,j)$ also present on the right hand side. This model has the advantage of being first order and can be obtained from G-R model easily by the following calculation. From 5 we obtain

$$x^h(i+1, j+1) = A_1(i, j+1)x^h(i, j+1) + A_2(i, j+1)x^v(i, j+1)$$
$$+ B_1(i, j+1)u(i, j+1)$$

$$x^v(i+1, j+1) = A_3(i+1, j)x^h(i+1, j) + A_4(i+1, j)x^v(i+1, j)$$
$$+ B_2(i+1, j)u(i+1, j)$$

which by considering $x(i, j) = [\ x^h(i, j), x^v(i, j)\ ]^T$ this equation set takes on the form 10 with the following matrices

$$J(i,j+1) = \begin{bmatrix} A_1(i,j+1) & A_2(i,j+1) \\ 0 & 0 \end{bmatrix}$$

$$K(i+1,j) = \begin{bmatrix} 0 & 0 \\ A_3(i+1,j) & A_4(i+1,j) \end{bmatrix}$$

$$E(i,j+1) = \begin{bmatrix} B_1(i,j+1) \\ 0 \end{bmatrix} \qquad F(i+1,j) = \begin{bmatrix} 0 \\ B_2(i+1,j) \end{bmatrix}$$

2.3.5 A-P Model

A new model for representation of the 2-D recursive digital filter was introduced by Alexander-Pruess [10]. This model has a pseudo-state representation with three coefficient matrices. This representation is very similar to new F-M model described by equation 10. These are actually equivalent by letting one of the coefficient matrices in the F-M model be equal to zero matrix. This representation has the following form

$$\begin{aligned} x(i+1,j+1) &= A_1(i+1,j)x(i+1,j) + A_2(i,j+1)x(i,j+1) + \\ &\quad B(i+1,j+1)u(i+1,j+1) \\ y(i+1,j+1) &= D(i+1,j+1)x(i+1,j+1) \end{aligned} \qquad (11)$$

$x(i+1,j+1)$ is a column vector such that its elements are the output $y(i-p,j-k)$ where $0 \le p \le M$ and $0 \le k \le M$.

2.3.6 F-T-R Model

Aravena and Porter [5] considered the state representation of m-D systems modeled as operators on a partially ordered Hilbert resolution space. They obtain a second order transition representation which is valid for both quarter plane and arbitrary conic causality structures and is called the fundamental transition relationship (F-T-R). The proposed model has the following format

$$
\begin{aligned}
x(i+1, j+1) = {} & J_{10}(i+1, j)x(i+1, j) + J_{01}(i, j+1)x(i, j+1) \\
& - K_{00}(i, j)x(i, j) + E_{10}(i+1, j)u(i+1, j) \\
& + E_{01}(i,j+1)u(i,j+1) - F_{00}(i,j)u(i,j). \qquad (12)
\end{aligned}
$$

Equation 12 is the most general form. By choosing $E_{01}(i,j+1) = E_{10}(i+1,j) = 0$ in above equation the F-M model is obtained. If $K_{00}(i,j) = F_{00}(i,j) = 0$ the model of form 10 is obtained.

Recently, in [75] it has been further established that every vector satisfying thr F-T-R evolution equation can be decomposed in the direct sum of two vectors. The two components can be described by a G-R type transition equation. The derivation of minimal G-R models has also been discussed in that work.

To summarize, the models of G-R, F-M, Attasi, modified F-M, and F-T-R are considered by many authors. There is a general preference for the G-R model. The basic reason is that the G-R model is first-order. On the other hand F-M, F-T-R and Attasi model are not first order and therefore x does not really satisfy a conventional state equation. To this point all these facts indicate that two dimensional state space models are complex and offers some difficult mathematical and conceptual problems. It appears that the best prospect for progress is to generalize one-dimensional concepts. For this reason we consider the 1-D Wave model established by Porter-Aravena [6] as equivalent to the stationary 2-D Roesser's model and extend it to nonstationary case.

2.4 THE EQUIVALENT 1-D STATE SPACE MODEL FOR 2-D SYSTEMS

The 1-D Wave model has been established by Porter-Aravena [6] as equivalent to the stationary 2-D state space models. In particular it was shown that the G-R and modified F-M models had an intrinsic 1-D behavior. When an m-D model evidences an intrinsic 1-D character several obvious advantages occur. Important among these is the possibility of transporting the very extensive 1-D system theory to the m-D setting. Similar to the stationary case the equivalent Wave model for the nonstationary 2-D systems can be constructed. This will be discussed in chapter 4. In the following sections we will present the equivalent Wave model for 2-D state space models. In the present study it suffices to recall the key definitions. A more complete discussion is available in [6].

2.4.1 Wave Model for Modified F-M Model

To formalize the observation let ϕ denote a column vector with entries computed in the following fashion:

$$\phi(0) = \mathrm{col}[x(0,0)]$$

$$\phi(1) = \mathrm{col}[x(1,0),\ x(0,1)]$$

$$\vdots$$

$$\phi(n) = \mathrm{col}[x(n,0),\ x(n-1,1),\ \ldots\ldots,\ x(0,n)].$$

Consider now the jth component of $\phi(n+1)$ namely,

$$\phi_j(n+1) = x(n+1-j,\ j), \qquad j = 0,1,\ldots..n. \tag{14}$$

Using 10 we have

$$\phi_j(n+1) = Jx(n-j,j) + Kx(n+1-j,j) + Eu(n-j,j) + Fu(n+1-j,j-1)$$

and using 14

$$\phi_j(n+1) = J\phi_j(n) + K\phi_{j-1}(n) + Ev_j(n) + Fv_{j-1}(n)$$

where

$$v(n) = \mathrm{col}[u(n,0),\ u(n-1,1),\ \ldots\ldots,\ u(0,n)].$$

Therefore the P-A model is as follows

$$\Phi(n+1) = A(n)\Phi(n) + B(n)v(n) \tag{15}$$

where

$$A(n) = \begin{bmatrix} J & 0 & 0 & 0 \cdots\cdots 0 \\ K & J & 0 & 0 \cdots\cdots 0 \\ 0 & K & J & 0 \cdots\cdots 0 \\ 0 & 0 & K & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \quad \vdots \\ \vdots & & & J \quad 0 \\ 0 & & & K \quad J \\ 0 & & & 0 \quad K \end{bmatrix}.$$

$$B(n) = \begin{bmatrix} E & 0 & 0 & 0 \cdots\cdots 0 \\ F & E & 0 & 0 \cdots\cdots 0 \\ 0 & F & E & 0 \cdots\cdots 0 \\ 0 & 0 & F & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \quad \vdots \\ \vdots & & & E \quad 0 \\ 0 & & & F \quad E \\ 0 & & & 0 \quad F \end{bmatrix}$$

The matrices and vectors of 15 expand as n increases. It should be noted that in the presence of boundaries equation 15 remains valid; however the top and bottom components of the ϕ tuplet become superfluous along some of the boundaries.

2.4.2 <u>Wave Model for G-R Model</u>

By a similar discussion as in the above section the G-R model can also be put in the Wave model.

Let $n \geq 0$ be an integer and define the tuplets

$\phi(n) = col(x^v(0,n),x^h(1,n-1),x^v(1,n-1),\ldots,x^v(n-1,1),x^h(n,0))$

$v(n) = col(u(0,n), \; u(1,n-1),\ldots,u(n,0))$

$\mu(n) = col(\mu(0,n), \; \mu(1,n-1),\ldots,\mu(n,0))$

$f(n) = col(x^h(0,n), \; x^v(n,0))$.

The dimension of these tuplets is obviously n dependent and in particular,

$$\dim \phi(n) = 2n \qquad\qquad \dim \mu(n) = n+1$$

$$\dim v(n) = n+1 \qquad\qquad \dim f(n) = 2$$

respectively.

Using the notation of above equations, one can consider the time variant difference equation,

$$\begin{aligned} \phi(n+1) &= A(n)\phi(n) + B(n)v(n) + E(n)f(n) \\ \mu(n) &= C(n)\phi(n) + D(n)v(n) + H(n)f(n). \end{aligned} \tag{16}$$

The matrices in question need to be of variable dimension, the various block dimension being:

$A(n) - (2n+2) \times 2n \qquad\qquad C(n) - (n+1) \times 2n$

$B(n) - (2n+2) \times (n+1) \qquad\qquad D(n) - (n+1) \times (n+1)$

$E(n) - (2n+2) \times 2 \qquad\qquad H(n) - (n+1) \times 2$

respectively.

In [6] it is shown that equation 16 is entirely equivalent to equation 1 provided the above matrices are chosen appropriately. In particular, the matrices

$$A(n) = \begin{bmatrix} A_4 & 0 & 0 & & & 0 \\ A_2 & 0 & 0 & & & 0 \\ 0 & A_3 & A_4 & & & \cdot \\ 0 & A_1 & A_2 & & & \cdot \\ & & & & & \\ & & & A_3 & A_4 & 0 \\ & & & A_1 & A_2 & 0 \\ 0 & & & & & A_3 \\ 0 & & & & & A_1 \end{bmatrix}$$

$$B(n) = \begin{bmatrix} B_2 & 0 & & 0 \\ B_1 & 0 & & 0 \\ 0 & B_2 & & \\ \cdot & B_1 & & \\ \cdot & & & \\ & & & \\ 0 & & & B_2 \\ 0 & & & B_1 \end{bmatrix}$$

$$C(n) = \begin{bmatrix} C^v & 0 & 0 \; . \; . \; . & & & & & 0 \\ 0 & C^h & C^v & & & & & \cdot \\ 0 & & & C^h & C^v & & & \cdot \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ & & & & & C^h & C^v & 0 \\ 0 & & & & & & & C^h \end{bmatrix}$$

$$E(n) = \begin{bmatrix} A_3 & 0 \\ A_1 & \cdot \\ 0 & \cdot \\ \cdot & \\ & 0 \\ 0 & A_4 \\ 0 & A_2 \end{bmatrix} \qquad H(n) = \begin{bmatrix} C^h & 0 \\ 0 & \cdot \\ \cdot & \cdot \\ & \\ & \\ 0 & C^v \end{bmatrix}.$$

The dimension of the matrices A(n), B(n), C(n), D(n), E(n), and H(n) are increasing as n increases. While D(n) = diag[...D...] renders equation 1 and 16 equivalent. The matrices A(n), B(n) and C(n) have the 'banded' structure discussed in [6] while D(n) is block diagonal and E(n), H(n) have two block columns.

2.4.3 Wave Model for F-T-R model

We now present the wave advance model for F-T-R model established by Aravena-Porter [5,75]. Our presentation is brief and the reader may find the references cited supply useful background.

Let $\phi(n)$ and $v(n)$ denote the following tuplets

$$\phi(n) = \mathrm{col}[x(n,0),\ x(n-1,1),\ \ldots\ldots x(0,n)]$$

$$v(n) = \mathrm{col}[u(n,0),\ u(n-1,1),\ \ldots\ldots u(0,n)] \qquad n = 0,1,\ldots.$$

It was shown [5] that the F-T-R model has the equivalent form

$$\phi(n+1) = J(n)\phi(n) + K^{-}(n-1)\phi(n-1) + E(n)v(n)$$
$$+ F^{-}(n-1)v(n-1) + \{.\}. \qquad (17)$$

The term {.} is due to boundary conditions and without loss of generality it can be assumed to be zero. The matrices $J(n)$, $K^{-}(n-1)$, $E(n)$, and $F^{-}(n-1)$ are defined as follows

$$J(n) = \begin{bmatrix} J_{01} & 0\ldots\ldots\ldots 0 & \\ J_{10} & 0\ldots\ldots\ldots 0 & \\ & \cdot & \\ & \cdot & J_{01} \\ 0\ldots\ldots\ldots & 0 & J_{10} \end{bmatrix}.$$

$$E(n) = \begin{bmatrix} E_{01} & 0 \cdots\cdots 0 \\ E_{10} & 0 \cdots\cdots 0 \\ & \ddots & \\ & & E_{01} \\ 0 \cdots\cdots 0 & & E_{10} \end{bmatrix}.$$

$$K^{-}(n-1) = I(n)K(n-1), \qquad F^{-}(n) = I(n)F(n-1)$$

$$K(n) = \mathrm{diag}[K_{00}],$$

$$F(n) = \mathrm{diag}[F_{00}]$$

$$I(n) = \begin{bmatrix} 0 \cdots & \cdots 0 \\ \vdots & \vdots \\ \vdots \quad I_n & \vdots \\ \vdots & \vdots \\ 0 \cdots & \cdots 0 \end{bmatrix}$$

where the dimension of above matrices are

$J(n)$ - $(n+2) \times (n+1)$ $\qquad$ $I(n)$ - $(n+2) \times n$

$E(n)$ - $(n+2) \times (n+1)$ $\qquad$ $F(n)$ - $(n+1) \times (n+1)$

$K(n)$ - $(n+1) \times (n+1)$.

The appearance of the term $\phi(n-1)$ on the right hand side of equation 17 reflects the fact that the F-T-R model is in fact a second order model. However, since we are dealing with a 1-D equation the conversion to a 1-D state model is immediate. By defining

$$\Upsilon(n) = K(n-1)\phi(n-1) + F(n-1)v(n-1)$$

the 1-D wave equation can be written in the following form

$$\begin{bmatrix} \phi(n+1) \\ \Upsilon(n+1) \end{bmatrix} = \begin{bmatrix} J(n) & I(n) \\ K(n) & 0 \end{bmatrix} \begin{bmatrix} \phi(n) \\ \Upsilon(n) \end{bmatrix} + \begin{bmatrix} E(n) \\ F(n) \end{bmatrix} v(n).$$

The technique presented above is a direct extension of conventional 1-D procedures. However, in [75] it has been shown that any vector satisfying an F-T-R equation can be decomposed into the direct sum of vectors satisfying a G-R model. Thus it is possible to derive a G-R model without increasing the dimensionality of the state vector.

2.4.4 Wave Model for F-M Model

It should be noted that the F-M model is a special case of F-T-R model. Actually if $E_{10} = E_{01} = 0$ then the F-T-R model is a F-M model. Thus substituting for the matrices E_{10} and E_{01} the value of zero the 1-D Wave model for F-M can be constructed as follows

$$\begin{bmatrix} \phi(n+1) \\ \Upsilon(n+1) \end{bmatrix} = \begin{bmatrix} J(n) & I(n) \\ K(n) & 0 \end{bmatrix} \begin{bmatrix} \phi(n) \\ \Upsilon(n) \end{bmatrix} + \begin{bmatrix} 0 \\ F(n) \end{bmatrix} v(n).$$

CHAPTER 3

STABILITY TEST FOR DIGITAL m-D FILTERS

3.1 INTRODUCTION

In this chapter selected stability results for stationary m-D recursive digital filters, (2-D in particular) are summarized. The space limitations preclude an exhaustive survey. We first state the definition of stability. We then cite only the most relevant theorems and state them without proofs. As a matter of style we first consider transfer function representation and state a result in the 2-D setting then extend this to m-D systems. Next we present stability analysis for 2-D state-space digital filters. We introduce also some basic notations for use here and in later chapters.

Def (1): A 2-D sequence $\{x(m,n)\}$ is said to be p-summable $1 \le p < \infty$ provided that

$$\sum_m \sum_n |x(m,n)|^p < \infty \quad \text{that is } \{x(m,n)\} \in l_p. \tag{1}$$

Remark. For the specific cases $p = 1$, $p = 2$ the terminology absolutely summable and square summable is used respectively.

Def (2): A 2-d sequence $\{x(m,n)\}$ is said to be bounded provided that there is a real constant $k < \infty$ such that

$$|x(m,n)| < k \qquad \forall\ m,n. \tag{2}$$

Def (3): A 2-D filter is bounded-input bounded-output (BIBO) stable provided that any input sequence $\{x(m,n)\} \in l_\infty$ produces an output sequence $\{y(m,n)\} \in l_\infty$.

It is well known that a 2-D filter with support in the first quadrant (+m,+n) is BIBO stable if and only if its impulse response h(m,n) is absolutely summable.

The following notation is introduced for convenience. The closed unit bidisk is defined as

$$\overline{U}^2 = \{(z_1,z_2)|\ |z_1| \le 1,\ |z_2| \le 1\}.$$

The open unit bidisk is defined as

$$U^2 = \{(z_1,z_2)|\ |z_1| < 1,\ |z_2| < 1\}.$$

The distinguished boundary of the unit bidisk is defined as

$$T^2 = \{(z_1,z_2)|\ |z_1| = |z_2| = 1\}.$$

The corresponding notation for n-D polydisk will be $\overline{U}^n$, U^n, and T^n respectively, for $n \ge 1$.

3.2 STABILITY ANALYSIS FOR 2-D RECURSIVE FILTERS

A major concern in the design of a 2-D digital filter is ensuring the stability of the filter. A widely used stability criterion is the so-called BIBO stability condition. The BIBO stability is ensured if the impulse response coefficients $h_{m,n}$ of the fiter are absolutely summable, i.e.,

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |h_{m,n}| < \infty.$$

This condition is automatically satisfied by a bounded

2-D FIR filter. This is due to the fact that it has only a finite number of nonzero impulse response coefficients. On the other hand, it is more difficult to ensure that the above condition holds for a 2-D IIR filter.

Considerable effort has been spent by several authors in developing stability theorem for IIR filters and formulating practical tests based on these theorems. A brief outline of the above mentioned theorems is stated in the following section.

Theorem (1) [11]: The two-dimensional system described by

$$H(z_1,z_2) = \frac{1}{Q(z_1,z_2)} \qquad (3)$$

is BIBO stable if and only if

$$Q(z_1,z_2) \neq 0 \quad \text{for all} \quad z_1,\ z_2 \in \bar{U}^2 \qquad (4)$$

Def (4): The polynomial $Q(z_1,z_2)$ is a Shanks' polynomial if $Q(z_1,z_2) \neq 0$ for all z_1, $z_2 \in \bar{U}^2$.

Remark. In the more general case if equation 3 has $P(z_1,z_2)$ as its numerator and no nonessential singularity of the second kind exists [12], and $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime two-dimensional polynomials,then the condition 4 is also the necessary and sufficient condition for stability. Theorem 1 was first given in [11] for the case $P(z_1,z_2)$ being nonunity. It was shown in [12] that the numerator plays an important role in the stability of the system. This is due to possible existence of the second kind of singularity and will be discussed later.

Several independent proofs of the above theorem are given in [13].

The above theorem has been simplified by several authors. The equivalent forms are as follows:

Theorem (2) [14]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z_1,0) \neq 0 \qquad |z_1| \leq 1$

ii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| \leq 1.$ (5)

The above theorem was also proved in [15], [16], [17], [18]. Conditions 5 are much easier to check than condition 4. Note that the role of z_1 and z_2 can be interchanged.

Theorem (3) [19]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z_1,b) \neq 0 \qquad |z_1| \leq 1$ for some b, $|b| = 1$

ii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| \leq 1.$ (6)

Theorem (4) [20]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z_1,b) \neq 0 \qquad |z_1| \leq 1$ for some b, $|b| = 1$

ii) $Q(a,z_2) \neq 0 \qquad |z_2| \leq 1$ for some a, $|a| = 1$

iii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| = 1.$ (7)

Theorem (5) [20]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z,z) \neq 0 \qquad |z| \leq 1$

ii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| = 1.$ (8)

Theorem (6) [20]: The 2-D system described by 3 is BIBO stable if and only if

ii) $Q(z^k,z^l) \neq 0 \quad |z| \leq 1$ where k and l are positive integers

iii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| = 1.$ (9)

Theorem (7) [21]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z_1,b) \neq 0 \qquad |z_1|\leq 1 \quad$ for some b, $b \in [0,1]$

ii) $Q(a,z_2) \neq 0 \qquad |z_2|\leq 1 \quad$ for some a, $|a| \leq 1$

iii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1$ and all $z_2 \in [0,1]$

iiii) $Q(z_1,z_2) \neq 0 \qquad |z_1| = 1, \quad |z_2| = 1. \qquad (10)$

Theorem (8) [22]: The 2-D system described by 3 is BIBO stable if and only if

i) $Q(z_1^0,z_2) \neq 0 \qquad |z_2|\leq 1, \ |z_1^0|\leq 1$

ii) $Q(z_1,z_2^0) \neq 0 \qquad |z_1|\leq 1 \quad |z_2^0| = 1$

iii) The equation $Q(z_1^{-1},z_2^{-1}) = 0$ has no solution for $|z_1| = |z_2| = 1. \qquad (11)$

We note that the proof of these stability theorems is approached differently by the several authors. Decarlo's and Strintzis's proofs are based on homotopy and Cauchy's principal value formula respectively. Rajan [23] proved the theorem using continuity argument and Woods [24] proved it using the residue theorem. We also mention that the Shank's conjecture [11] and theorem 1 of [25], about the relation of BIBO stability and planar least squares inverse have been proven to be wrong [26]. In [27-32] the application of theorem [22] is given. In all above theorems the filter is assumed to be first-quadrant causal.

For a more general region U the following theorems are available.

Theorem (9) [21]: The 2-D polynomial $Q(z_1,z_2) \neq 0$ in $\bar{U}^2$ if and only if

i) $Q[f_1(z_1), f_2(z_2)] \neq 0$ in $z_1 \leq 1$

where $f_1(z_1)$ and $f_2(z_2)$ are some continuous mapping of $\bar{U}^1$ into $\bar{U}^1$ and of T^1 into T^1 with Ind $f_1(e^{j\theta}) > 0$, Ind $f_2(e^{j\theta}) > 0$, $0 \leq \theta \leq 2\pi$

ii) $Q(z_1, z_2) \neq 0$ on T^2. (12)

<u>Theorem (10) [26]</u>: If we assume that $\{Q(m,n)\}$ has finite support, there exists a BIBO stable filter given by $1/Q(z_1,z_2)$ whose smallest region of support is a symmetric half-plane if and only if

$$
\begin{array}{ll}
\text{i)} & Q(z_1,z_2) \neq 0 \quad \text{on} \quad T^2 \\
\text{ii)} & Q(1,z_2) \neq 0 \quad |z_2| \leq 1 \\
\text{iii)} & Q(z_1,1) \neq 0 \quad |z_1| \leq 1.
\end{array}
\tag{13}
$$

We now state the BIBO stability in terms of the impulse response. It is known that the impulse response $\{h(n)\}$ of a 1-D filter described by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^n \tag{14}$$

is BIBO stable if and only if, one of the following equivalent statements are satisfied:

i) $$\lim_{n \to \infty} |h(n)| = 0 \tag{15}$$

ii) $$\sum_{n=0}^{\infty} |h(n)|^{1/p} < \infty \quad \text{for any } p \geq 1 \tag{16}$$

iii) $$\lim_{n \to \infty} \sup |h(n)|^{1/n} < 1 \tag{17}$$

iiii) $$|h(n)| \leq c r^n \quad 0 \leq c < \infty, \quad 0 < r < 1. \tag{18}$$

Goodman [12] has shown that the above statements are not necessarily true for two and multidimensional digital filters. For instance, it is shown that the filter is unstable but $\{h(m,n)\} \in l_2$ and also $\lim h(m,n) = 0$ as m and n go to infinity. The relation of BIBO stability and impulse response is given in the following:

$$Q(z_1,z_2) \neq 0 \quad \text{in } U^2 \overset{\not\Longleftarrow}{\Longrightarrow} \text{BIBO stability}$$

$$Q(z_1,z_2) \neq 0 \quad \text{in } U^2-T^2 \overset{\Longleftarrow}{\not\Longrightarrow} \text{BIBO stability}$$

$$\{h(m,n)\} \in l_1 \Longleftrightarrow \text{BIBO stability}$$

$$\{h(m,n)\} \in l_2 \overset{\Longleftarrow}{\not\Longrightarrow} \text{BIBO stability}$$

$$\lim_{m,n \to \infty} h(m,n) \to 0 \overset{\Longleftarrow}{\not\Longrightarrow} \text{BIBO stability}$$

$$Q(z_1,z_2) \neq 0 \quad \text{in } U^2 \overset{\Longleftarrow}{\not\Longrightarrow} |h(m,n)| \leq k < \infty, \ \forall\, m,n$$

$$|H(z_1,z_2)| \leq k < \infty \ \text{in } U^2 \overset{\not\Longleftarrow}{\Longrightarrow} \{h(m,n)\} \in l_2$$

$$Q(z_1,0) \neq 0 \quad |z_1| \leq 1 \overset{\not\Longleftarrow}{\Longrightarrow} \sum_{n=0}^{\infty} |h(m,n)| < \infty, \ \forall\, m$$

$$Q(0,z_2) \neq 0 \quad |z_2| \leq 1 \overset{\not\Longleftarrow}{\Longrightarrow} \sum_{m=0}^{\infty} |h(m,n)| < \infty, \ \forall\, n$$

3.3 <u>STABILITY OF MULTIDIMENSIONAL SYSTEMS</u>

The algebraic conditions in theorem 1-6 can be easily modified for multidimensionality. However, the affiliated computational requirements are much more substantial. The m-D generalization of the above theorems are as follows

<u>Theorem (11)[33]</u>: The n-dimensional first-quadrant recursive digital filter is stable if and only if

$$Q(z_1,z_2,\ldots\ldots,z_n) \neq 0 \qquad \text{on} \quad \bigcap_{i=1}^{n} |z_i| \leq 1. \tag{19}$$

Theorem (12)[33]: The n-dimensional first-quadrant recursive digital filter is stable if and only if

$$Q(z_1,0,\ldots\ldots,0) \neq 0 \quad \text{on} \quad |z_1| \le 1$$

$$Q(z_1,z_2,0\ldots\ldots,0) \neq 0 \quad \text{on } \{|z_1| = 1\} \cap \{|z_2| \le 1\}$$

$$Q(z_1,z_2,\ldots\ldots,z_{n-1},0) \neq 0 \quad \text{on } \{\bigcap_{i=1}^{n-2} |z_i| = 1\} \cap \{|z_{n-1}| \le 1\}$$

$$Q(z_1,z_2,\ldots\ldots,z_n) \neq 0 \quad \text{on } \{\bigcap_{i=1}^{n-1} |z_i| = 1\} \cap \{|z_n| \le 1\}. \quad (20)$$

Theorem (13) [19]: The n-dimensional first-quadrant recursive digital filter is stable if and only if

i) $Q(z_1,z_2,\ldots\ldots,z_n) \neq 0$ on $|z_1|=|z_2|=\ldots|z_n|=1$

ii) For some $b_1,\ldots.b_n$ such that $|b_r| = 1$, $r = 1,2,\ldots,n$ and for all i, $i = 1,2,\ldots.,n$

$$Q(z_1,z_2,\ldots,z_n) \neq 0 \text{ when } z_r = b_r,\ r \neq i, \text{ and } |z_i| \le 1. \quad (21)$$

The above theorem is proved independently also in [26].

Theorem (14) [20]: The n-dimensional first-quadrant recursive digital filter is stable if and only if

i) $Q(z,z,\ldots\ldots,z) \neq 0$ on $|z| \le 1$.

ii) $Q(z_1,z_2,\ldots,z_n) \neq 0$ on $|z_1|=|z_2|=\ldots.|z_n|=1$. (22)

The above theorem is also established in [16].

Theorem (15) [21]: The n-dimensional first-quadrant recursive digital filter is stable if and only if

i) $Q(T^n) \neq 0$

ii) $Q(z,\ldots..z) \neq 0$ in U. (23)

The generalization of theorem 6 is stated by the same authors in [22].

3.4 STABILITY TEST

In order to check the stability of a 2-D system we are required to check the following conditions:

$$Q(z_1,z_2) \neq 0 \qquad \text{on} \quad |z_1| \leq 1, \ |z_2| \leq 1 \tag{24}$$

or

$$Q(z_1,z_2) \neq 0 \qquad \text{on} \quad |z_1| = 1, \ |z_2| \leq 1 \tag{25}$$

or

$$Q(z_1,z_2) \neq 0 \qquad \text{on} \quad |z_1| = 1, \ |z_2| = 1. \tag{26}$$

Note that it is also necessary to check the stability of the one-dimensional system (i.e. verify condition i) of theorems 2-5. There are many algebraic methods based on the inners, division method, or the table form for the latter case and are discussed in [34,35].Clearly it is much easier to check condition 26 or 25 than 24. The above conditions are easily extended to n-dimensional case.

There exists two different procedures in order to check the above conditions: algebraic and mapping(or numerical). The first procedure consists of mainly the following three methods:

1) determinant method [36,37]
2) resultant method [29,38]
3) table form [39,40,41].

The second procedure contains the root mapping [11], the Nyquist test [20], the cepstral method [42], and the method based on impulsive response methods. We briefly state some of the above methods.

3.4.1 Determinant Method

In [36,37] the condition 26 is considered. The polynomial $Q(z_1,z_2)$ is assumed to be one-dimensional with coefficients as a function of variable z_1. The Schur-Cohn test is applied. In this method one is required to check the positive definiteness of a Hermitian matrix $C(z_1,z_2)$ (reciprocal of $Q(z_1,z_2)$). It has been shown [37] that the test of determinant $[C(z_2)] > 0$ on $|z_2| = 1$ and $C(z_1^0)$ is required for establishing positive definiteness. The test can carried out using Sturm sequences by change of variable, or Cohn's theorem on reciprocal polynomials . The congruence method of computing the determinant is the most efficient way in this case.

3.4.2 Resultant Method

This method has been discussed in [43] for the one dimensional case. Zeheb and Walach [22] have modified this for the multidimensional case. In this method, a single variable polynomial $p(x)$ is constructed from the set of n equations in such a way that if there exists a real solution to the set of equations, then there also exists a real zero of $p(x)$. Hence, if $p(x)$ has no real zeros, condition 24 is satisfied. If $p(x)$ has k real zeros, then those values have to be substituted into the set of n equations, to get k problems of dimension $n-1$. The process continues successively.

3.4.3 Table Form

Maria and Fahmy [39] have modified the Jury table to check the roots of polynomials with complex coefficients.

The modified table is used to check the condition $Q(z_1,z_2) = 0$ on $|z_1| \leq 1$, $|z_2| = 1$ by considering

$$Q(z_1,z_2) = \sum_{n=0}^{M} a_n(z_2)z_1^n \tag{27}$$

where $a_n(z_2)$ is a polynomial in z_2. This developement results in the following theorem.

<u>Theorem (16) [39]</u>: $Q(z_1,z_2) \neq 0$ on $|z_1| \leq 1$, $|z_2| = 1$, if and only if

$$b_0 > 0,\ c_0 > 0, \ldots\ldots\ldots, t_0 > 0$$

where b_0 , c_0 ,.........,t_0 are obtained from the modified Jury table.

There are also papers that consider other algebraic methods. For instance Bose and Basu [44] stated a method based on Rudins's theorem, and Kayran and King [45] discussed a technique based on the inners determinant. Other algebraic methods are considered by Huang [14] and Ansell [46]. For a mapping test see [11,20,42]. The various mapping methods are compared numerically in [2]. The above methods are extended to multidimensional systems by Jury [47], Bose [48].

3.5 <u>SOME NECESSARY AND SUFFICIENT STABILITY CONDITIONS FOR LOW-ORDER 2-D SYSTEMS</u>

In general, it is very difficult, if not impossible, to obtain the algebraic stability conditions for two and multidimensional systems in terms of equation coefficients. This is not surprising since this difficulty exists also for high order one-dimensional cases. As a consequence the stability conditions are stated in terms of a positive-definite matrix or a positive-innerwise matrix [47]. However, explicit conditions can be obtained up to the quartic equation case [49]. In a similar form, explicit conditions have been obtained for low-order 2-D systems.

Case 1 [50]

Given $Q(z_1,z_2) = 1 + az_1 + bz_2$. The system is BIBO stable if and only if

$$|a| + |b| < 1.$$

The above polynomial can be generalized to

$$Q(z_1,z_2) = 1 + az_1^k z_2^p + bz_1^m z_2^n$$

provided that $kn - pm = 0$.

Case 2 [50]

Given $Q(z_1,z_2) = 1 + az_1 + bz_2 + cz_1z_2$. The system is BIBO stable if and only if

$$|a + b| - 1 < c < 1 - |a - b|.$$

The above polynomial can be generalized to

$$Q(z_1,z_2) = 1 + az_1^k z_2^p + bz_1^m z_2^n + cz_1^{km} z_2^{pn}$$

provided that $kn - pm = 0$.

Case 3 [50]

Given

$$Q(z_1,z_2) = 1+az_1+bz_2+cz_3+dz_1z_2+ez_2z_3+fz_3z_1+gz_1z_2z_3.$$

The system is BIBO stable if and only if

$$|a| < 1, \quad |(1 - a)/(b - d)| > 1, \quad |(1 + a)/(b + d)| > 1$$

$$A < 0, \quad B < 0, \quad C < 0, \quad E < 0, \quad D^2 < 4BC + 4AE + 8(ABCE)^{1/2}$$

where

$$A = (c - e - f + g)^2 - (1 - a - b + d)^2$$

$$B = (c + e - f - g)^2 - (1 - a + b - d)^2$$

$$C = (c - e + f - g)^2 - (1 + a - b - d)^2$$

$$D = 8(d + fe - ab - cg)$$

$$E = (c + e + f + g)^2 - (1 + a + b + d)^2.$$

3.6 SOME SUFFICIENT CONDITIONS FOR STABILITY

From the preceding section, it is clear that testing the stability conditions for 2-D, and m-D systems is tedious. Therefore it is appropriate to look for some easier conditions even though those conditions are only sufficient.

Reddy [51] stated a sufficient condition. This condition was obtained from the Walach and Zebeb [32] conditions. Another sufficient condition was obtained by Chiasson and Brierly [52]. This is not as simple as the condition stated in [51]. Some sufficient conditions for instability are stated in [53]. Some important results from the efforts are given in the following.

Theorem (17) [51]: The n-dimensional digital filter polynomial

$$Q(z_1,z_2,\ldots z_k) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots\cdots \sum_{i_k=0}^{n_k} \alpha_{i1,\ldots ik}\, z_1^{i1}\ldots z_k^{ik} \neq 0$$

in U^n if

$$\alpha_{0,0,\ldots 0} > \Sigma|\alpha_{i1,\ldots ik}|, \qquad i_1 \in R_1,\ldots i_k \in R_k$$

and $(i_1 + i_2 + \ldots + i_k) \neq 0$, where $\alpha_{0,\ldots 0}$ is assumed to be positive.

Theorem (18) [53]: $Q(z_1,z_2)$ is not a Shanks' polynomial if one of the following inequalities is satisfied.

i) For real values of z_1, z_2 and $0 < z_1 \le 1$, $|z_2| \ge 1$,

$$Q(z_1,z_2) \ge (1 + |z_1|)^n (1 + |z_2|)^m$$

$$Q(z_1,z_2) \le (1 - |z_1|)^n (1 + |z_2|)^{m-1}(1-|z_2|).$$

ii) For real values of z_1, z_2 and $0 < z_1 \le 1$, $0 < z_2 \le 1$,

$$Q(z_1,z_2) \ge (1 + |z_1|)^n (1 + |z_2|)^m$$

$$Q(z_1,z_2) \le (1 - |z_1|)^n (1 - |z_2|)^m \qquad (28)$$

where

$$Q(z_1,z_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} q_{ij} z_1^i z_2^j, \qquad q_{00} = 1.$$

<u>Theorem (19) [53]</u>: A polynomial $Q(z_1,z_2)$ is unstable if

$$\sum_{i=0}^{n} q^2_{ip} > \binom{m}{p} \sum_{i=0}^{n} q^2_{i0} \qquad p = 1,2,\ldots,m$$

or

$$\sum_{i=0}^{m} q^2_{pi} > \binom{n}{p} \sum_{i=0}^{m} q^2_{0i} \qquad p = 1,2,\ldots,n \qquad (29)$$

<u>Theorem (20) [53]</u>: Let

$$Q(z_1,z_2,\ldots z_k) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots\cdots \sum_{i_k=0}^{n_k} b_{i1,\ldots,ik}\, z_1^{i_1} \cdots z_k^{i_k}$$

and $b_{0\ldots0} = 1$. The polynomial is unstable if one of the following inequalities is satisfied.

i) For real values of z_i, and $0 < z_i \le 1$, $i = 1,\ldots,k$,

$$Q(z_1,\ldots z_k) \ge (1 + |z_1|)^{n_1} \cdots (1 + |z_k|)^{n_k}$$

$$Q(z_1,\ldots z_k) \le (1 - |z_1|)^{n_1} \cdots (1 - |z_k|)^{n_k}.$$

ii) For real values of z_i, and $|z_i| \ge 1$, $i = 1,\ldots,k$,

$$0 < |z_i| \le 1 \quad i = 1,\ldots,k, \quad i \ne 1$$

$$Q(z_1,\ldots z_k) \ge (1 + |z_1|)^{n_1} \cdots (1 + |z_k|)^{n_k} \qquad (30)$$

$$Q(z_1,\ldots z_k) \le (1-|z_1|)^{n_1} \cdots (1+|z_1|)^{n_1-1} \cdots (1-|z_k|)^{n_k}.$$

Theorem (21) [54]: Let

$$Q(z_1,z_2,\ldots z_n) = \sum_{i=1}^{M} \alpha_i \, z_1^{k_{i1}} \ldots\ldots z_n^{k_{in}} + K$$

where the matrix $[k_{ij}]$ is of rank equal to the number of monomials m, not including a possible constant term. Then $Q(z_1,z_2,\ldots z_n)$ is stable if and only if

$$|K| > \sum_{i=1}^{M} |\alpha_i|. \tag{31}$$

3.7 THE EFFECT OF NONESSENTIAL SINGULARITIES OF THE SECOND KIND FOR THE STABILITY

So far we have discussed the stability of the system presented by

$$H(z_1,z_2) = \frac{1}{Q(z_1,z_2)}.$$

Since the numerator was chosen to be equal to one, nonessential singularities of the second kind (see Def 2.2) were avoided. However if the numerator is an arbitrary polynomial $P(z_1,z_2)$ we have a different situation. This is because unlike the 1-D case the numerator effects an important role on the stability of 2-D systems. Note that if $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually coprime [55,56] with no nonessential singularities of the second kind, the conditions imposed for the case $P(z_1,z_2) = 1$ constitute the necessary and sufficient condition for stability. However, in the presence of such singularities the conditions are only sufficient for BIBO stability [12].

3.8 TESTING NONESSENTIAL SINGULARITIES OF THE SECOND KIND

Some methods are presented for testing the existence of non essential singularities of the second kind. In [57] the problem of existence is translated to determining local positivity of a real polynomial in n real variables. Walach and Zeheb [58] have also presented two alternative methods which are computationally simpler than the method in [57]. We now state one of these methods.

Theorem (22) [58]: Let

$$H(z_1,\ldots\ldots z_n) = \frac{P(z_1,\ldots\ldots z_n)}{Q(z_1,\ldots\ldots z_n)} \qquad (32)$$

be a real function in n variables. Assume $n \leq 4$ and $P(z)$ and/or $Q(z)$ are not symmetric. Then equation 32 has nonessential singularities of the second kind on the distinguished boundary of the unit polydisk if and only if

$$P(z) = 0, \quad Q(z) = 0, \quad P(z^{-1}) = 0, \quad \text{and} \quad Q(z^{-1}) = 0$$

has a solution, where

$$z = (z_1,z_2\ldots\ldots\ldots z_n) \text{ and } z^{-1} = (z_1{}^{-1},\ldots\ldots\ldots z_n{}^{-1}).$$

In [59] a sufficient condition for BIBO stability when $H(z_1,z_2)$ has nonessential singularity of the second kind is discussed.

3.9 STABILITY ANALYSIS FOR 2-D STATE-SPACE DIGITAL FILTERS

Recent progress on internal descriptions of 2-D systems has provided the possibility of describing the stability question in a state-space format. An extended Lyapunov theory has been developed for checking the stability of the system. The 2-D Lyapunov equation was first presented by Piekarski [60] for 2-D continuous systems. Lodge-Fahmy [61] extended this to the discrete case using the double bilinear transformation. Unfortunately it has been shown in [62] that the 2-D Lyapunov condition is in general only sufficient for stability and not necessary. A new verifiable test for stability was developed by Lu-Lee [63] based on Lyapunov theory. Some of these results will be presented in the following.

Def (5): The stationary system of equation 1.5 is said to be asymptotically stable if whenever $u(n) = 0$, $\lim x(n,m) = 0$ as n, and/or m goes to ∞ for initial conditions

$$x^v(n,0) = 0, \qquad n \geq N$$
$$x^h(0,m) = 0, \qquad m \geq M$$

Theorem (23) [60]: The two variable characteristic polynomial $f(s) = \det[(s_1 I_n + s_2 I_m) - B]$ is a two dimensional Hurwitzian polynomial if and only if there exists a positive definite Hermitian matrix $G = G_n \oplus G_m$ such that

$$W = GB \oplus B^T G$$

is negative definite where $\oplus$ is direct sum of matrices.

The following theorem is the extension of the above theorem for real 2-D discrete systems.

Theorem (24) [62]: The 2-D characteristic polynomial $g(z^{-1}) = \det[(z_1{}^{-1} I_n \oplus z_2{}^{-1} I_m) - A]$ is a Shank's polynomial if there exists a positive definite symmetric matrix $G = G_n \oplus G_m$ such that

$$W = G - A^T G A \tag{33}$$

is positive definite.

It should be noted that the above theorem was originally stated in [61] for necessary and sufficient condition. Unfortunately in [62] it was shown that the above theorem holds only for sufficient part.

<u>Theorem (25) [64]</u>: A stationary 2-D system having a state-space model as in 2.5, with initial conditions such that

$$x^v(n,0) = 0, \qquad n \geq N$$
$$x^h(0,m) = 0, \qquad m \geq M$$

is asymptotically stable if there exists a positive definite matrix $G = G_n \oplus G_m$ such that

$$W = G - A^T G A$$

is positive definite.

The polynomial $P(z_1,z_2)$ given in 2.6 can be rewritten in the following factor form

$$P(z_1,z_2) = |z_1 I_n - A_1| \cdot |z_2 I_m - [A_4 + A_3(z_1 I_n - A_1)^{-1} A_2]|$$
$$= |z_2 I_m - A_4| \cdot |z_1 I_n - [A_1 + A_2(z_2 I_m - A_4)^{-1} A_3]|.$$

The following theorem is equivalent to the theorem 2.

<u>Theorem (26) [65]</u>: The following statements are equivalent:

1) the stationary system 2.5 is BIBO stable;

2) i) A_1 is stable,

   ii) $A_4 + A_3(z_1 I_n - A_1)^{-1} A_2$ with $|z_1|$ is stable; (34)

3) i) A_4 is stable,

   ii) $A_1 + A_2(z_2 I_m - A_4)^{-1} A_3$ with $|z_2|$ is stable; (35)

4) i) A is stable,

ii) A_1 has no eigenvalues on the unit circle,

iii) $A_4 + A_3(z_1 I_n - A_1)^{-1} A_2$ with $|z_1|$ has no eigenvalues on the unit circle; (36)

5) i) A is stable,

ii) A_4 has no eigenvalues on the unit circle,

iii) $A_1 + A_2(z_2 I_m - A_4)^{-1} A_3$ with $|z_2|$ has no eigenvalues on the unit circle. (37)

Corollary (1) [65]: The following three conditions are necessary for BIBO stability of the stationary system 2.5:

1) A is stable;

2) A_1 is stable;

3) A_4 is stable.

Theorem (27) [65]: The following conditions are sufficient for the BIBO stability of the stationary system 2.5

i) A_1 and A_4 are stable,

ii) A_1 and A_4 are diagonalizable and the transformation

$$\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = T \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

is chosen such that $\hat{A}_1$ and $\hat{A}_4$ are diagonal, where $A = TAT^{-1}$,

iii) $||\hat{A}_2||\,||\hat{A}_3|| < (1 - e)(1 - s)$ (38)

where

$e = \max|\sigma(A_1)|$ $\qquad$ $s = \max|\sigma(A_4)|$

$\sigma(.)$ is the eigenvalue of a matrix.

Alexander-Pruess [10] introduced some theorems that will be stated now.

<u>Theorem (28) [10]</u>: The discrete linear shift invariant system represented by equation 2.11 is BIBO stable if and only if for at least one matrix norm the following equation holds

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ||h(m,n)|| < \infty$$

where $h(m,n) = DC(A_1{}^m A_2{}^n)BH_{00}$. The term $C(A_1{}^j A_2{}^k)$ is equal to the sum of all product terms involving all permutations of A_1 as a factor, j times and A_2 as a factor, k times and H_{00} is the input vector which presents a single unit sample at the (0,0) position of 2-D data array with all other input samples zero.

<u>Theorem (29) [10]</u>: The discrete linear shift invariant system represented by equation 2.11 and for which the numerator and denominator polynomials of the corresponding transfer function are mutually prime is unstable if any one of the $\sigma_{max}(A_1)$, $\sigma_{max}(A_2)$, or $\sigma_{max}(A_1+A_2)$ is greater than or equal to one.

CHAPTER 4

STABILITY ANALYSIS OF THE WAVE MODEL

ABSTRACT

This chapter considers the 1-D Wave model established by Porter-Aravena [6] as an equivalent to the nonstationary 2-D Givone-Roesser (G-R) model [2]. The Wave model provides a new approach to the stability problem for 2-D digital recursive filters. Stability tested in multi-step is defined. Several theorems are stated which express sufficient conditions for stability in terms of the spectral radius of the system matrices. In particular, necessary and sufficient conditions are stated for checking the stability of 2-D systems for a special case. Application of Lyapunov theory to the Wave model is also discussed. Several examples are given to demonstrate the development.

4.1 INTRODUCTION

An important goal in the design of a 2-D digital filter is to ensure the stability of the device. In chapter 3 the transfer function description of a 2-D system was considered and the BIBO stability criterion was discussed (see [66] and [67] for more details). In that chapter several techniques and results from a number of authors were summarized.

This chapter continues the study of 2-D stability. Our interest here is with Wave model format that will be introduced shortly. The Wave model has both conceptual and practical advantages. The most important of the wave model is that multidimensional equations are cast in single dimensional form. As such, appropriate single dimensional results are potentially transportable to the multidimensional setting. In [6] it is shown that every conic causal map has a wave model. Here we consider the G-R model. For the reader's convenience we repeat first the conventions refered to as the G-R model for a nonstationary 2-D discrete system namely:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} B_1(i,j) \\ B_2(i,j) \end{bmatrix} u(i,j) \tag{1}$$

$$y(i, j) = \begin{bmatrix} c_1(i,j) & c_2(i,j) \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}$$

where $x^h \varepsilon R^n$, $x^v \varepsilon R^m$ represent the horizontal and vertical local states respectively, u is the input and y the output. The system matrix A is given by

$$A(i,j) = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix}$$

with the submatrices A_k, $k=1,\ldots,4$ of appropriate dimensions.

The introduction of state-space models for 2-D systems suggests using stability techniques common to state-space formats. In the 1-D case, these consist primarily of spectral methods, the Lyapunov theory, and norm bounds on the state transition matrix. In [65] theorems were established for testing the stability of a stationary 2-D system by the spectral approach. The tests involved, however, included conditions which are not easily verified (for example condition 2(ii) or 3(ii) of theorem 3.1). This is due to the fact that two-variable polynomials of finite order may have an infinite number of roots. Furthermore for the nonstationary case the spectral approach does not determine the stability. The 2-D Lyapunov method could be an important tool for the stability analysis in the state space. Unfortunately, it has been shown [62] that the 2-D Lyapunov condition is, in general, only sufficient for stability and not necessary. However, nonstationary 2-D systems are untouched. In this chapter we consider the state space model for nonstationary 2-D systems.

Our development uses to advantage a system model, namely the Wave model, recently introduced in [6]. This model has the advantage of being 1-D. In [6] it is also shown that all quarter plane causal maps have a Wave model. The key to such a conversion is the concept of a Wave model. In the present study it suffices to recall the key definitions. A more complete discussion is available in chapter 2.

Let $n \geq 0$ be an integer and define the tuplets

$$\phi(n) = \mathrm{col}(x^v(0,n),\ x^h(1,n-1),\ldots,x^v(n-1,1),x^h(n,0))$$

$$v(n) = \mathrm{col}(u(0,n),\ u(1,n-1),\ldots,u(n,0))$$

$$\mu(n) = \mathrm{col}(y(0,n),\ y(1,n-1),\ldots,y(n,0))$$

$$f(n) = \mathrm{col}(x^h(0,n),\ x^v(n,0)).$$

The dimension of these tuplets is obviously n dependent and in particular,

$$\dim \phi(n) = 2n \qquad \dim \mu(n) = n+1$$

$$\dim v(n) = n+1 \qquad \dim f(n) = 2$$

respectively.

Using the notation of above equations, one can consider the time variant difference equation,

$$\phi(n+1) = A(n)\phi(n) + B(n)v(n) + E(n)f(n)$$

$$\mu(n) = C(n)\phi(n) + D(n)v(n) + H(n)f(n). \qquad (2)$$

The matrices in question need to be of variable dimension, the various block dimensions being:

$A(n)$ - $(2n+2) \times 2n$ $\qquad$ $C(n)$ - $(n+1) \times 2n$

$B(n)$ - $(2n+2) \times (n+1)$ $\qquad$ $D(n)$ - $(n+1) \times (n+1)$

$E(n)$ - $(2n+2) \times 2$ $\qquad$ $H(n)$ - $(n+1) \times 2$

respectively.

In [6] it is shown that equation 2 is entirely equivalent to equation 1 provided the above matrices are chosen appropriately. In particular, the matrices

$$A(n) = \begin{bmatrix} A_4(n,0) & 0 & 0 & \cdots & & 0 \\ A_2(n,0) & 0 & 0 & \cdots & & 0 \\ 0 & A_3(n-1,1) & A_4(n-1,1) & & & 0 \\ 0 & A_1(n-1,1) & A_2(n-1,1) & & & \\ & & \ddots & & & \\ & & A_3(1,n-1) & A_4(1,n-1) & & 0 \\ & & A_1(1,n-1) & A_2(1,n-1) & & 0 \\ 0 & & & & & A_3(0,n) \\ 0 & & & & & A_1(0,n) \end{bmatrix}$$

$$B(n) = \begin{bmatrix} B_2(n,0) & 0 & 0 \\ B_1(n,0) & 0 & 0 \\ 0 & B_2(n-1,1) & \\ \cdot & B_1(n-1,1) & \\ \cdot & & \ddots \\ 0 & & B_2(0,n) \\ 0 & & B_1(0,n) \end{bmatrix}$$

$$C(n) = \begin{bmatrix} C^v(n,0) & 0 & 0 & & 0 \\ 0 & C^h(n-1,1) & C^v(n-1,1) & & \cdot \\ 0 & & \ddots & & \\ \cdot & & & & \\ \cdot & & & & \\ & & C^h(1,n-1) & C^v(1,n-1) & 0 \\ 0 & & 0 & 0 & C^h(0,n) \end{bmatrix}$$

$$E(n) = \begin{bmatrix} A_3(n,0) & 0 \\ A_1(n,0) & \cdot \\ 0 & \cdot \\ \cdot & \\ & 0 \\ 0 & A_4(0,n) \\ 0 & A_2(0,n) \end{bmatrix} \qquad H(n) = \begin{bmatrix} C^h(n,0) & 0 \\ 0 & \cdot \\ \cdot & \cdot \\ & \cdot \\ 0 & C^v(0,n) \end{bmatrix}$$

while $D(n) = \mathrm{diag}[\ldots D \ldots]$ render equation 1 and 2 equivalent. The matrices $A(n)$, $B(n)$ and $C(n)$ have the 'banded' structure discussed in [6] while $D(n)$ is block diagonal and $E(n)$, $H(n)$ have two block columns.

While the stability of equation 2 is a 1-D issue, the dimension of the state, ϕ, expands with time, and the state transition matrix $A(n)$ is not square. Since $A(n)$ is not square one can not use an eigenvalue analysis of this matrix to check the stability. Therefore stability is approached utilizing a matrix norm.

4.2 STABILITY DEFINITIONS

For conciseness we begin with the following definitions, both of which refer to the system of equation 2 with $v(.) = 0$ and $f(.) = 0$. We shall use the term asymptotically stable and marginally stable for the two properties in question.

Def (1): The system of equation 2 is asymptotically stable provided that for every k and finite $||\phi(k)||$ the sequence $||\phi(n+k)||$ goes to zero as n goes to ∞.

Def (1'): The system of equation 2 is marginally stable provided that for every k and finite $||\phi(k)||$ the inequality $||\phi(n+k)|| \leq M$ holds for some $M < \infty$ and all $n > 0$.

The next several theorems establish sufficient conditions for stability and marginal stability respectively. These conditions all arise from the iterative character of equation 2, namely

$$\phi(n+k+1) = A(n+k)A(n+k-1).......A(k)\phi(k) \qquad (3)$$

and the companion norm inequality

$$||\phi(n+k+1)|| \leq ||A(n+k)||.||A(n+k-1)||....||A(k)||.||\phi(k)|| \qquad (4)$$

Grouping matrices by adjacent pairs we have also

$$||\phi(n+k+1)|| \leq ||A(n+k)A(n+k-1)||.....||A(k+1)A(k)||. ||\phi(k)||. \qquad (5)$$

Similar and obvious inequalities arise when grouping by adjacent triplets, etc. is applied.

Theorem (1): If, for every k, $\prod_{i=k}^{p} ||A(i)||$ goes to zero as p goes to ∞, then the system of equation 2 is asymptotically stable.

Proof: Using the inequality of 4 and definition 1 the theorem is evident.■

Theorem (1'): If, for every k, $\prod_{i=k}^{p} ||A(i)|| \leq M < \infty$ as p goes to ∞, then the system of equation 2 is marginally stable.

Proof: Using the inequality of 4 and definition 2 the theorem is evident.■

The following theorems state alternative conditions for assuring the stability.

Theorem (2): If $||A(i)|| < 1-\varepsilon$ for $i = 1,2,3,\ldots.$ where $\varepsilon > 0$, then the system of equation 2 is asymptotically stable.

Proof: Since $||A(i)|| < 1-\varepsilon$ it follows from the inequality of 4 that $||\phi(n+k)|| \leq (1-\varepsilon)^n$. Thus $||\phi(n+k)||$ goes to zero as n goes to ∞.■

It should be noted that if the theorem 2 holds then the theorem 1 also holds.

Theorem (2'): If $||A(i)|| \leq 1$ for $i = 1,2,3,\ldots.$ then the system of equation 2 is at least marginally stable.

Proof: With $||A(i)|| \leq 1$ we have $||\phi(n+k)|| \leq ||\phi(k)||$ which with $M = ||\phi(k)||$ satisfies the marginal stability criteria.■

Theorem (3): If, for every k, $\prod_{i=k}^{p} ||A(2i+1)A(2i)||$ goes to zero as p goes to ∞ then the system of equation 2 is asymptotically stable.

Proof: Similar to the proof of theorem 1 but using the inequality 5.■

Theorem (3'): If, for every k, $\prod_{i=k}^{p} ||A(2i+1)A(2i)|| \leq M < \infty$ as p goes to ∞ then the system of equation 2 is marginally stable.

Proof: Similar to the proof of the theorem 1'.■

Theorem (4): If $||A(2i+1)A(2i)|| < 1-\varepsilon$ for all $i \geq k$, some finite k and some $\varepsilon > 0$ then the system of equation 2 is asymptotically stable.

Proof: Similar to the proof of theorem 2 using inequality 5.■

Theorem (4'): If $||A(2i+1)A(2i)|| \leq 1$ for all $i \geq k$, some finite k, then the system of equation 2 is at least marginally stable.

Proof: Similar to the proof of theorem 2'.■

The last four theorems utilize inequality 5 and are refered to as 2-step tests. There are m-step extensions of these criteria which would involve the function

$$\varsigma(k) = ||A(mk+k-1)........A(mk)||.$$

In fact theorems 3, 3', 4, 4' hold with $\varsigma(k)$ replacing $||A(2i+1)A(2i)||$ in the statement of these theorems.

As a perspective on the multistep tests we note that the norm generally suppresses any algebraic structure that the A(n) might possess. To illustrate, consider the case where the A(n) are square, stationary, and nilpotent, but with $||A(n)|| > 1$. The 1-step test is inconclusive. However, the m-step test (m being the nilpotency of A) verifies the stability of the system. We shall present a family of staionary G-R model, parametrized by the integer $k > 0$, which has inconclusive m-step tests for $m < k$ but has a conclusive k-step test.

Concerning all of the above theorems, any true norm will suffice. While all such norms are topologically equivalent the specific criteria of the above theorems may be met by the l_2 norm but not the l_1, for example.

In the subsequent analysis we utilize the l_2 norm on R^n. In this norm it is known that $||A||_2$ = [max eigenvalue of $A^TA]^{1/2}$. The fact that A^TA is square even when A is not is attractive in the wave advance model setting. Indeed we shall soon see that $A^T(n)A(n)$ has a block diagonal form which further facilitates the search for $||A(n)||$.

4.3 COMPUTING NORMS

The utility of theorems 1, 2, 3, 4 of section 4.2 depends explicitly on the calculation of matrix norms, for example $||A(n)||$. Since the matrices A(n) have a dimensionality which increases with n, it is not apriori clear that the requisite calculations are viable. We shall see, however, that the structure of the A(n) can be made to yield useful results.

4.3.1 Computing $||A(n)||$

In our developement we shall tacitly imply an l_2 norm on R^n unless stated otherwise. In this regard consider $\Phi(n) = [A(n)]^T[A(n)]$ where A(n) is specified in equation 2. It is readily verified that $\Phi(n)$ has the block diagonal form.

$$\Phi(n) = \begin{bmatrix} M(n,0) & 0 & 0 & \cdots & & 0 \\ 0 & Q(n-1,1) & 0 & & & \\ & & Q(n-2,2) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & Q(1,n-1) & 0 \\ 0 & 0 & & & & N(0,n) \end{bmatrix} \qquad (6)$$

where

$$Q(1,k) = \begin{bmatrix} N(1,k) & K(1,k) \\ K^T(1,k) & M(1,k) \end{bmatrix} \tag{7}$$

$$M(1,k) = A^T{}_4(1,k)A_4(1,k) + A^T{}_2(1,k)A_2(1,k)$$

$$N(1,k) = A^T{}_1(1,k)A_1(1,k) + A^T{}_3(1,k)A_3(1,k)$$

$$K(1,k) = A^T{}_3(1,k)A_4(1,k) + A^T{}_1(1,k)A_2(1,k)$$

In the interests of brevity we adopt a few standard terminologies and notations. The spectrum of the matrix T is denoted by $\sigma(T)$. The matrices of interest here are Hermitian and positive, hence each $\Gamma \in \sigma(T)$ is real and nonnegative. The maximum eigenvalue is denoted by

$$\sigma_{max}(T) = \max \{\Gamma: \Gamma \in \sigma(T)\}.$$

Our first result is an immediate consequence of the block diagonal character of $\Phi(n)$.

Result (1): The spectrum of matrix $\Phi(n)$ is equal to the union of the spectrum of the matrices $M(n,0)$, $N(0,n)$, and $Q(n-i,i)$ for $i = 1,....n-1$.

From the existing literature [68] we lift the following theorem.

Theorem (5): Let A be a n-square Hermitian matrix with characteristic roots $\alpha_1 \geq \ldots \geq \alpha_n$.
Let B be a k-square principal submatrix of A with characteristic roots $\beta_1 \geq \ldots \geq \beta_k$, then

$$\alpha_s \geq \beta_s \geq \alpha_{n-k+s}, \quad s = 1,\ldots,k.$$

Noting that $M(n,0)$, $N(0,n)$ are principal submatrices of $Q(n,0)$ and $Q(0,n)$ respectively, the above theorem implies that $\sigma_{max}Q(n,0) \geq \sigma_{max}M(n,0)$ and $\sigma_{max}Q(0,n) \geq \sigma_{max}N(0,n)$ for all n. Using these observations we have the following lemma.

Lemma (1): For $\Phi(n)$ of equation 6 and $Q(l,k)$ of equation 7

$$\sigma_{max} [\Phi(n)] = \max_i \{\sigma_{max}Q(n-i,i): i=0,...n\}.$$

Several special cases are suggested by the equality of lemma 1. The most obvious is the case where the coefficient matrix $A(i,j)$ is stationary. It follows that $Q(n-i,i)$ is then stationary and

$$\sigma_{max}[\Phi(n)] = \sigma_{max}[Q]. \tag{8}$$

Thus $||A(n)||$ is independent of n.

Furthermore if the matrix A is symmetric then the above equation is equivalent to

$$\sigma_{max}[\Phi(n)] = \sigma_{max}{}^2[A]. \tag{9}$$

As a second example we assume that the coefficient matrix $A(i,j)$ is stationary along each wave front, $i+j = n$. It follows easily that $Q(n-i,i)$ is independent of i, thus

$$\sigma_{max}[\Phi(n)] = \sigma_{max}[Q(n)]. \tag{10}$$

A further simplification occurs when the coefficients are wave front periodic, that is $A(i,j) = A(i',j')$ whenever $i' + j' = i + j + T$. We then have

$$\sigma_{max}[\Phi(n)] = \sigma_{max}[\Phi(n+T)] = \sigma_{max}[Q(n)] \tag{11}$$

with calculations necessary only for $n = 1,....,T$.

4.3.2 Computing $||A(n)A(n-1)||$

In the analysis of $||A(n)||$ we have retained the generality of nonstationary equations. The multistep analysis, however, is motivated by the goal of reclaiming some of the algebraic characteristics of $A(n)$ in the norm criteria. Restriction of attention to the stationary case supports this objective and, moreover, greatly simplifies the notation. Thus we invoke stationarity throughout the present section.

For reasons discussed in section 4.3.1 we continue with the l_2 norm. Using an obvious modification of earlier notation we define

$$\Phi(n;1) = \Phi(n) = A^T(n)A(n)$$

$$\Phi(n;2) = A^T(n)A^T(n+1)A(n+1)A(n)$$

$$\vdots$$

$$\vdots$$

$$\Phi(n;k) = A^T(n)....A^T(n+k)A(n+k)....A(n)$$

in which case

$$||A(n+k)...A(n)||^2 = \sigma_{max}[\Phi(n;k)].$$

Consider now the matrices $A(n)$ of equation 2. By direct examination it is readily verified that the matrices $A(n+k)...A(n)$ have a banded structure. It follows also that the matrices $\Phi(n;k)$ have a symmetric banded structure. A more complete discussion is available in appendix A. Our interest initially is with $\Phi(n;2)$ which is summarized in the following.

The matrix $\Phi(n;2)$ is the $(2n) \times (2n)$ block banded matrix specified by

$$\Phi(n;2) = \begin{bmatrix} X & Y & Z & 0 & 0 & 0 & 0 & \cdots & 0 \\ Y^T & X & Y & Z & 0 & 0 & 0 & \cdots & 0 \\ Z^T & Y^T & X & Y & Z & 0 & 0 & \cdots & 0 \\ 0 & Z^T & Y^T & X & Y & Z & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & Z^T & Y^T & X \end{bmatrix}$$

where

$$X = \begin{bmatrix} G & H \\ H^T & F \end{bmatrix}$$

$$Y = \begin{bmatrix} L & 0 \\ S & R \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix}$$

The block matrices composing X, Y, and Z are given by

$$G=(A_4{}^2)^T A_4{}^2+(A_2A_4)^T(A_2A_4)+(A_3A_2)^T(A_3A_2)+(A_1A_2)^T(A_1A_2)$$

$$H=(A_3A_2)^T(A_4A_3)+(A_1A_2)^T(A_2A_3)$$

$$F=(A_1{}^2)^T A_1{}^2+(A_3A_1)^T(A_3A_1)+(A_2A_3)^T(A_2A_3)+(A_4A_3)^T(A_4A_3)$$

$$L=(A_3A_2)^T A_4{}^2+(A_1A_2)^T(A_2A_4)$$

$$S=(A_4A_3)^T A_4{}^2+(A_2A_3)^T(A_2A_4)+(A_3A_1)^T(A_3A_2)+(A_1{}^2)^T(A_1A_2)$$

$$R=(A_3A_1)^T A_4{}^3+(A_1{}^2)^T(A_2A_3)$$

$$P=(A_3A^1)^T A_4{}^2+(A_1{}^2)^T(A_2A_4)$$

In section 4.3.1 we found that for stationary systems

$$\sigma_{max}[\Phi(n;1)] = \sigma_{max}[Q]$$

and hence the norm calculation $||A(n)||$ was independent of n. This stationary result is not available for $\Phi(n;k)$ for $k > 1$. Thus σ_{max} $[\Phi(n;2)]$ would need to be computed for each n individually. While several efficient algorithms exist (see [69] and [70] for example) to achieve this, the specific structure of $\Phi(n;2)$ does not seem to assist in any way.

The structure of $\Phi(n;2)$ does, however, facilitate establishing an upper bound for σ_{max} $[\Phi(n;2)]$ which is independent of n. We now introduce two approaches for obtaining such an upper bound. The first approach takes advantage of an algebraic theorem [71] which bounds the maximum eigenvalue in terms of the elements of the matrix. The second approach uses the results available for the circulant matrices.

4.3.3 Algebraic Approach

For the reader's convenience we state a portion of the algebraic theorem in question [71].

Theorem (6) [71]: Let $T=(t_{ij})$ be any arbitrary p x p matrix, and let

$$\Omega_i = \sum_{j=1, j\neq i}^{n} |t_{ij}| \quad 1 \le i \le n,$$

then all the eigenvalues of T lie in the union of the disks

$$|z-t_{ii}| \le \Omega_i \quad 1<i<n.$$

Theorem (6') [71]: If $T=(t_{ij})$ is an arbitrary p x p matrix and

$$\delta = \max_{1\le j\le n} \sum_{i=1}^{p} |t_{ij}|,$$

then

$$|\sigma_{max}(T)| \le \delta.$$

Theorem (6'') [71]: If $T = (t_{ij})$ is an arbitrary p x p matrix and

$$\delta = \max_{1\le j\le n} \sum_{i=1}^{p} |t_{ji}|,$$

then

$$|\sigma_{max}(T)| \le \delta.$$

Remark 1. Note that if theorem 6 is satisfied then theorems 6' and 6'' are also satisfied. This is due to the fact that the disk $|z-t_{ii}| < \Omega_i$ is a subset of the disk $|z| \le |t_{ii}|+\Omega_i$.

Using the above theorems give the following results for the banded matrix $\Phi(n;2)$.

Corollary (1): If $\delta = \max\{\delta_1 , \delta_2\}$ where

$$\delta_1 = \max_{1\le j\le n} \left[\sum_{i=1}^{n} \{|r_{ij}|+|h_{ij}|+|f_{ij}|+|s_{ji}|+|r_{ji}|+|p_{ji}|\} \right]$$

and

$$\delta_2 = \max_{1\le j\le n} \left[\sum_{i=1}^{n} \{|p_{ij}|+|l_{ij}|+|s_{ij}|+|g_{ij}|+|h_{ji}|+|l_{ji}|\} \right],$$

then $\sigma_{max}[\Phi(n;2)] \le \delta$.

Proof: We know that the matrix $\Phi(n;2)$ is a tridiagonal matrix. Therefore taking the maximum value of the matrix $\Phi(n;2)$ row-wise is the same as taking the maximum of the matrix $\Phi(3;2)$ row-wise. Using theorem 6' yields the results.■

In the above corollary the maximum was taken column-wise in determining an upper bound for the spectral radius of the matrix $\Phi(n;2)$. Since $\Phi(n;2)$ is symmetric the bound can be determined via an analogous computation row-wise.

Corollary (2): Let $\Gamma_3 = \max\{g_{jj}, f_{jj} \quad j=1,2,...n\}$ and

$$\Gamma_4 = \underset{1\le j\le n}{\mathrm{Max}} \{ \delta_3=\delta_1-f_{jj} \quad , \delta_4=\delta_2-g_{jj}\}.$$

Then all eigenvalues of $\Phi(n;2)$ are between $(\Gamma_3-\Gamma_4)$ and $(\Gamma_3+\Gamma_4)$.

Proof: The corollary is a direct result of theorem 6.■

4.3.4 Circulant Approach

In this section we establish an upper bound for the eigenvalues of $\Phi(n;k)$, $k = 2,...$. For this approach we introduce a family of block circulant matrices $C_k(n)$. We show that $\sigma_{max}[\Phi(n;k)] \le \sigma_{max}[C_k(n)]$ and, in addition, that $\sigma_{max}[C_k(n)]$ is independent of n. Thus while $\sigma_{max}[\Phi(n;k)]$ is wave step dependent our bound criteria is not.

Consider then the block circulant matrix

$$C(n) = \begin{bmatrix} X_1 & X_2 & \cdots\cdots & & X_n \\ X_n & X_1 & X_2 & \cdots & X_{n-1} \\ & & & \cdot & \\ & & & & \cdot \\ X_2 & X_3 & X_4 & \cdots & X_1 \end{bmatrix}$$

where $X_1,....,X_n$ are square matrices of equal dimension. To diagonalize C(n) the scalar valued functions

$\phi_i(k) = \exp[j2\pi(i-1)k/n] \quad i = 1,2,....,n \quad k = 0,1....n-1.$

will be useful. The matrices $\mu(k)$ then defined by

$$\mu(k) = \phi_1(k)X_1 + \phi_2(k)X_2 + \cdots\cdots \phi_n(k)X_n. \tag{12}$$

The block column matrices

$$W(k) = \text{col-block } [I\phi_1(k):I\phi_2(k): \;.....\; :I\phi_n(k)]. \tag{13}$$

and the subsequent block matrix

$$W = [W(0):W(1):........:W(n-1)]$$

will also be necessary. It is readily verified that

$$W^{-1} = \text{block } 1/n[I\phi_1(-k):I\phi_2(-k): \ldots\ldots :I\phi_n(-k)] \tag{14}$$

where $\phi_i(-k) = \phi_i^*(k)$ ('*' denotes conjugate transpose) should be noted.

Lemma (2): For D = diag-block $[\mu(0):\mu(1):\ldots\ldots\mu(n-1)]$ and the matrices $C(n)$, W and W^{-1} defined earlier,

$$C(n) = WDW^{-1}.$$

Proof: This lemma may be verified by inspection. The format $W^{-1}C(n)W$ is perhaps the most direct approach.∎

Our interest lies with the special case $C(n) = C^*(n)$. By inspection we see $X_1 = X_1^*$, $X_2 = X_n^*$, ... $X_j = X_{n+2-j}{}^*$, $j = 3, \ldots, n/2$. It should also be noted that $\phi_n(k) = \phi_2^*(k)$, $\phi_{n-1}(k) = \phi_3^*(k)$, and so on. Then $\mu(k)$ takes the form

$$\mu(k) = \phi_1(k)X_1 + \phi_2(k)X_2 + \ldots\ldots + \phi_3^*(k)X_3^* + \phi_2^*(k)X_2^*. \tag{15}$$

From lemma 2 we conclude

$$\sigma_{max}\,[C(n)] = \max\,\{\sigma_{max}\mu(k): \; k = 0,1,\ldots,n-1\}. \tag{16}$$

Theorem (7): For $C(n) = C^*(n)$ and $\text{Re}(X_i) \geq 0$ the following inequality chain holds

$$\begin{aligned}\sigma_{max}\,[\mu(k)] &= ||\mu(k)|| = ||\mu(0)|| \\ &= ||X_1 + (X_2+X_2^*) + \ldots\ldots\ldots + (X_{n/2}+X_{n/2}^*)|| \\ &\leq ||X_1||+2||X_2||+2||X_3||+\ldots\ldots 2||X_p||\end{aligned}$$

where even n is assumed for simplicity.

Proof: The first equality holds since the matrix $\mu(k)$ is Hermitian. On the other hand assuming X_i are positive semi-definite yields

$$||\mu(k)|| = \sup_{||\alpha||=1} \langle\alpha \quad \mu(k)\alpha\rangle$$

$$= \sup_{||\alpha||} \{\langle\alpha \ X_1\alpha\rangle + \cos(2\pi k/n)\langle\alpha \ (X_2 + X_2^*)\alpha\rangle + \ldots.\}$$

$$= ||X_1 + (X_2+X_2^*) + (X_3+X_3^*) + \ldots. \qquad ||$$

$$= ||\mu(0)||$$

The last inequality holds by the norm properties and by the fact that the eigenvalues of a given matrix and its conjugate transpose are identical.∎

By combining the result of above theorem and equation 16 we obtain a upper bound for C(n). Thus the following result is immediate.

Result (2): By considering the matrix C(n) defined earlier we have

$$\sigma_{max}\,[C(n)] \leq \quad ||X_1||+2||X_2||+2||X_3||+\ldots\ldots 2||X_{n/2}||.$$

We now consider two special cases. For the first case we assume $X_i = 0$ for $i = 3,4,\ldots.,n-1$. Thus the new matrix, namely $C_1(n)$ with $\mu_1(k)$ and $W_1(k)$ as eigenmatrices has the following form

$$C_1(n) = \begin{bmatrix} X_1 & X_2 & 0 & 0 & 0 & \ldots\ldots.0 & X_2^T \\ X_2^T & X_1 & X_2 & 0 & 0 & \ldots\ldots.0 & 0 \\ 0 & X_2^T & X_1 & X_2 & 0 & \ldots\ldots.0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & X_2 \\ X_2 & 0 & 0 & \ldots\ldots\ldots.0 & & X_2^T & X_1 \end{bmatrix}$$

Considering $C_1(n)$, the above result can be modified as follows.

Result (3): By considering the matrix $C_1(n)$ defined earlier we have

$$\sigma_{max}\ [C_1(n)] \le ||X_1|| + 2||X_2||.$$

We now want to obtain an upper bound for the matrix $\Phi(n;1)$ of section 4.3.1. It is evident that $\Phi(n;1)$ can be written in an alternative form as follows

$$\Phi(n;1) = \begin{bmatrix} X_1 & X_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ X_2^T & X_1 & X_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & X_2^T & X_1 & X_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & X_2 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & X_2^T & X_1 \end{bmatrix}$$

where

$$X_1 = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}$$

and M, N, and K are defined in section 3.3.1.

It is apparent that

$$C_1(n) = \begin{bmatrix} & & & & X_2^T \\ & & & & 0 \\ & & \Phi(n;1) & & \vdots \\ & & & & 0 \\ & & & & X_2 \\ X_2 & 0 & \cdots\cdots 0 & X_2^T & X_1 \end{bmatrix}$$

Since $\Phi(n;1)$ is a principal submatrix of $C_1(n)$, then by theorem 5 we can obtain an upper bound for $\Phi(n;1)$. Thus we have the following lemma.

Lemma (3): For the matrices $\Phi(n;1)$, and $C_1(n)$ defined earlier we have

$$\sigma_{max}\ [\Phi(n;1)] \le \sigma_{max}\ [C_1(n)] \le ||X_1|| + 2||X_2||.$$

For the second case we assume $X_1 = X$, $X_2 = Y$, $X_3 = Z$ and

$$X_i = 0 \quad \text{for} \quad i = 4,5,\ldots\ldots\ldots,n-2$$

and X, Y, and Z are as defined in section 4.3.2.

This matrix, namely $C_2(n)$ has the form

$$C_2(n) = \begin{bmatrix} X & Y & Z & 0 & 0 & \cdots & 0 & Z^T & Y^T \\ Y^T & X & Y & Z & 0 & 0 & 0 & \cdots & Z^T \\ Z^T & Y^T & X & Y & Z & 0 & 0 & \cdots & 0 \\ 0 & Z^T & Y^T & X & Y & Z & 0 & \cdots & 0 \\ \cdots & & & & & & & & \cdots \\ \cdots & & & & & & & & \cdots \\ \cdots & & & & & & & & \cdots \\ Z & 0 & \cdots & 0 & Z^T & Y^T & X & & Y \\ Y & Z & 0 & 0 & \cdots & 0 & Z^T & Y^T & X \end{bmatrix}$$

It is also apparent that

$$C_2(n) = \left[\begin{array}{cccccc|cc} & & & & & & Z^T & Y^T \\ & & & & & & 0 & Z^T \\ & & & & & & : & 0 \\ & & & \Phi(n;2) & & & : & : \\ & & & & & & 0 & : \\ & & & & & & Z & 0 \\ & & & & & & Y & Z \\ Z & 0 & \cdots & 0 & Z^T & Y^T & X & Y \\ Y & Z & 0 & \cdots & 0 & Z^T & Y^T & X \end{array}\right].$$

By a similar discussion similar to the first case we can determine an upper bound for eigenvalues of the matrix $\Phi(n;2)$. Thus the following lemma is self evident.

<u>Lemma (4)</u>: For the matricies $\Phi(n;2)$, $C_2(n)$, and $\mu_2(k)$ defined earlier we have

$$\sigma_{max}\ [\Phi(n;2)] \le \sigma_{max}\ [C_2(n)] = \sigma_{max}\ [\mu_2(k)]$$
$$\le ||X|| + 2||Y|| + 2||Z||$$

for all $k = 0,1,\ldots,n-1$ and any n.

Considering the first case we even can determine the maximum eigenvalue of the matrix $C_1(n)$ for all n by the following manipulations. Substituting for X_1 and X_2 in equation 12 yields

$$\mu_1(k) = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} \exp[j2\pi k/n] + \begin{bmatrix} 0 & K^T \\ 0 & 0 \end{bmatrix} \exp[-j2\pi k/n]$$

$$= \begin{bmatrix} M & K^T \exp(-j2\pi k/n) \\ K \exp(j2\pi k/n) & N \end{bmatrix} .$$

We now state the following lemmas.

<u>Lemma (5)</u>: The spectrum of $\mu_1(k)$ is the same as the spectrum of $\mu_1(0)$ for all $k = 0,1,2,\ldots, n-1$ and any n.

<u>Proof</u>: This is due to the fact that $\mu_1(k) = P\ \mu_1(0)\ P^{-1}$ where

$$P = \begin{bmatrix} \exp(-j2\pi k/n)I & 0 \\ 0 & I \end{bmatrix} .$$

Moreover, we know that the spectrum is invariant under a similarity transformation.■

Combining the result of the above lemma with equation 16 yields

$$\sigma_{max}\Phi(n;1) \le \sigma_{max}C_1(n) = \sigma_{max}\mu_1(k) = \sigma_{max}\mu_1(0) = \sigma_{max}C_1(0)$$
$$= \sigma_{max}\Phi(1;1)$$

where

$$C_1(0) = \Phi(1;1) = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_1 \end{bmatrix}.$$

On the other hand by theorem 5 $\sigma_{max}\Phi(1;1) \leq \sigma_{max}\Phi(n;1)$. Thus $\sigma_{max}\Phi(n;1) = \sigma_{max}\Phi(1;1) = \sigma_{max}C_1(0) = \sigma_{max}\mu_1(0)$. From section 4.3.1 we also know that $\sigma_{max}\Phi(n;1) = \sigma_{max}Q$ for all n. In order to see how $\sigma_{max}Q$ and $\sigma_{max}\mu_1(0)$ are related we state the following lemma.

Lemma (6): The spectrum of $\mu_1(0)$ is the same as the spectrum of Q defined in section 4.3.1.

Proof: This is because of the similarity transformation given below

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} M & K^T \\ K & N \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} N & K \\ K^T & M \end{bmatrix} = Q. \blacksquare$$

In summary we have the following result.

Result (4): For the matrices $\Phi(n;1)$, $C_1(0)$, $\mu_1(k)$, and Q defined previously we have

$$\sigma_{max}\,[\Phi(n;1)] = \sigma_{max}\,[C_1(0)] = \sigma_{max}\,[\mu_1(0)] = \sigma_{max}\,Q.$$

4.4 STABILITY ANALYSIS

In section 4.2 sufficient conditions for asymptotic stability (respectively marginal stability) were embodied in theorems 1, 2, 3, 4 (respectively 1', 2', 3', 4'). These conditions utilized norm criteria on the matrices A(n). In section 4.3 the determination of norms and/or bounds on norms for matrices with the special structure of A(n) was considered.

In the present section the first order of business is to merge the results of sections 4.2 and 4.3 into concise statements of stability criteria. The section then takes up some special cases with affiliated modifications of these criteria.

4.4.1 1-Step

We consider then the system of equation 2 and the matrices A(i,j), A(n), $\Phi(n;k)$ and Q(1,k) affiliated with it. Recall that $\sigma_{max}(T)$ denotes the maximum eigenvalue of a Hermitian matrix T. For convenience we introduce the notation

$$\text{(i)} \quad \bar{\sigma}_Q(n) = \max_j \{\sigma_{max}{}^{1/2} Q(n-j,j) : j=0,1,..,n\}. \qquad (17)$$

Recall from section 4.3.1 that the matrix Q has the form

$$Q(i,j) = \begin{bmatrix} A_3(i,j) & A_4(i,j) \\ A_1(i,j) & A_2(i,j) \end{bmatrix}^T \begin{bmatrix} A_3(i,j) & A_4(i,j) \\ A_1(i,j) & A_2(i,j) \end{bmatrix}$$

The coefficient matrix A(i,j) of the G-R model has the form

$$A(i,j) = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix}.$$

We note that the component matrix used in forming Q(i,j) differs from A(i,j) by a row permutation. Moreover, it can be readily verified that

$$Q(i,j) = A^T(i,j)A(i,j) .$$

Thus we have

$$\sigma_{max}{}^{1/2}Q(n-j,j) = ||A(n-j,j)||$$

while

$$\text{(ii)} \quad \bar{\sigma}_Q(n) = \max_j \{||A(n-j,j)|| : j=0,1,..,n\}. \tag{18}$$

One special case is also noted. If the matrix A(i,j) is itself Hermitian for all (i,j) then

$$||A(i,j)|| = |\sigma_{max} A(i,j)|$$

Hence

$$\text{(iii)} \quad \bar{\sigma}_Q(n) = \max_j \{|\sigma_{max} A(n-j,j)| : j=0,1,..,n\}. \tag{19}$$

In the two following theorems we use the symbol $\bar{\sigma}_Q(n)$ in a generic sense, leaving open the choice of computational formula as appropriate to the properties of the matrix A(i,j) in question.

We turn now to the task of coalescing the results of sections 4.2 and 4.3. For brevity we shall use the phrase 'the system' in lieu of the more accurate 'the system of equation 2 with all external inputs zero'.

Using lemma 1 and theorem 2 we have

Theorem (8):

(i) If for every $k \le p$

$$\lim_{p \to \infty} \prod_{n=k}^{p} \bar{\sigma}_Q(n) = 0$$

then the system is asymptotically stable.

(ii) If there exists $M \in R$ such that, for every k

$$\prod_{n=k}^{\infty} \bar{\sigma}_Q(n) \le M < \infty$$

then the system is marginally stable.

Using lemma 1 together with theorem 2 we have:

Theorem (9):

(i) If there exists $\varepsilon > 0$ and integer $\bar{n}$ such that

$$\bar{\sigma}_Q(n) \le 1-\varepsilon, \qquad \text{all } n \ge \bar{n}$$

then the system is asymptotically stable.

(ii) If $\bar{\sigma}_Q(n) \le 1, \qquad \text{all } n \ge \bar{n}$

then the system is marginally stable.

Consider now the case where the coefficient matrices of the G-R model are stationary. Even though A(n) is nonstationary the analysis of section 4.3.2 has established that $||A(n)||$ is constant. Similarly Q(i,j) is constant and consequently $\bar{\sigma}_Q(n)$ is constant. For stationary A(i,j) we have then

$$\bar{\sigma}_Q = \sigma_{max}{}^{1/2}\ Q = ||A|| \tag{20}$$

and if A is symmetric

$$\bar{\sigma}_Q = |\sigma_{max}(A)|. \tag{21}$$

It is apparent then that 'stationary' permits a further simplification of theorems 8 and 9. Moreover we note that the product

$$\prod_{n=k}^{p} ||A||$$

goes to zero if and only if $||A|| < 1$ and that this product is bounded if and only if $||A|| \le 1$. Thus theorems 8 and 9 are equivalent in the stationary case.

We have then the result

<u>Theorem (10)</u>: For stationary A the system is

(i) asymptotically stable provided $\bar{\sigma}_Q < 1$,

(ii) marginally stable provided $\bar{\sigma}_Q \le 1$

One other refinement is also possible. When A is symmetric $||A|| = |\sigma_{max}(A)|$ and theorem 10 can be strengthened to necessary and sufficiency. For completeness we embody this refinement in a corollary.

<u>Corollary (3)</u>: For stationary symmetric A the system is

(i) asymptotically stable iff $|\sigma_{max}(A)| < 1$

(ii) marginally stable iff $|\sigma_{max}(A)| \le 1$

4.4.2 <u>2-Step</u>

In this section we restrict our attention to the stationary case. It can then easily be verified that theorems 3 and 4 (respectively 3' and 4') are equivalent. The modified form is as follows.

<u>Theorem (11)</u>: The system is asymptotically stable if $||A(2i+1)A(2i)|| < 1$ for all i.

<u>Theorem (11')</u>: The system is marginally stable if $||A(2i+1)A(2i)|| \le 1$ for all i.

As was discussed in section 4.3.2 for the general case, the norm $||A(n+1)A(n)||$ depends on n. Therefore σ_{max} $[\Phi(n;2)]$ can not be determined in a finite number of steps as n goes to ∞. However in sections 4.3.3 and 4.3.4 we have shown that there exists an upper bound for the eigenvalues of $\Phi(n;2)$.

Considering the results of section 4.3.3 the following theorems can be stated.

Theorem (12):

(i) If $\delta < 1$ then the system is asymptotically stable.

(ii) If $\delta \leq 1$ then the system is marginally stable.

where $\delta = \max\{\delta_1, \delta_2\}$ and

$$\delta_1 = \underset{1\leq j\leq n}{\text{Max}} \left[\sum_{i=1}^{n} \{|r_{ij}|+|h_{ij}|+|f_{ij}|+|s_{ji}|+|r_{ji}|+|p_{ji}|\} \right]$$

$$\delta_2 = \underset{1\leq j\leq n}{\text{Max}} \left[\sum_{i=1}^{n} \{|p_{ij}|+|l_{ij}|+|s_{ij}|+|g_{ij}|+|h_{ji}|+|l_{ji}|\} \right]$$

Theorem (13):

(i) If $(\Gamma_3+\Gamma_4)<1$ then the system is asymptotically stable.

(ii) If $(\Gamma_3+\Gamma_4)\leq 1$ then the system is marginally stable.

where Γ_3 and Γ_4 are

$$\Gamma_3 = \text{Max}\{g_{jj}, f_{jj} \quad j=1,2,\ldots n\} \text{ and}$$

$$\Gamma_4 = \underset{1\leq j\leq n}{\text{Max}} \{\delta_3=\delta_1-f_{jj} \quad , \delta_4=\delta_2-g_{jj}\}.$$

Due to the result of section 4.3.4 the following theorem is self-evident.

Theorem (14):

(i) If $||X||+2||Y|+2||Z|| < 1$ then the system is asymptotically stable.

(ii) If $||X||+2||Y||+2||Z|| \leq 1$ then the system is marginally stable.

where X, Y, and Z are defined in section 4.3.2.

4.5 BLOCK TRIANGULARITY

We consider here the case where the state transition matrix 'A' of the G-R model is block triangular (i.e. $A_2 = 0$ or $A_3 = 0$, or both). In this case the 2-D equation separates into two 1-D equations and stability can then be determined by the following lemma.

Lemma (7): The system equation 1 with $A_2 = A_3 = 0$ is asymptotically stable if and only if

$$|\sigma_{max}(A_1)| < 1 \quad \text{and} \quad |\sigma_{max}(A_4)| < 1. \tag{22}$$

In this section we consider the norm criteria. We superimpose the block triangularity assumption on A. The resultant specialization of norm criteria is then compared with lemma 7.

Case 1 ($A_2 = A_3 = 0$). In this case the matrices $\Phi(n;1)$, $\Phi(n;2)$, ...$\Phi(n;i)$... will have the following form:

$$\Phi(n;1) = \text{diag}\,[A_4^TA_4,\ A_1^TA_1]$$

$$\Phi(n;2) = \text{diag}\,[(A_4^2)^T(A_4^2),\ (A_1^2)^T(A_1^2)]$$

:

:

$$\Phi(n;i) = \text{diag}\,[(A_4^i)^T(A_4^i),\ (A_1^i)^T(A_1^i)].$$

Then by the use of definition 1 the following lemma is immediate.

Lemma (8): For the case $A_2 = A_3 = 0$ the system is asymptotically stable if

$$\sigma_{max}\,[(A_4{}^i)^T(A_4{}^i)] < 1 \quad \text{and} \quad \sigma_{max}\,[(A_1{}^i)^T(A_1{}^i)] < 1 \qquad (23)$$

for some i.

The following theorem shows that the condition 23 is also necessary.

Theorem (15): The following statements are equivalent. For the case $A_2 = A_3 = 0$

i) The system is asymptotically stable (24)

ii) $|\sigma_{max}(A_1)| < 1$ and $|\sigma_{max}(A_4)| < 1$ (25)

iii) $||A_1{}^k|| < 1$ and $||A_1{}^k|| < 1$ for some k. (26)

Proof: Part i) and ii) are equivalent by lemma 7. We now want to show ii) and iii) are equivalent.

iii-->ii Condition 26 implies that $|\sigma_{max}(A_1{}^k)| < 1$, and $|\sigma_{max}(A_4{}^k)| < 1$. Therefore $|\sigma_{max}(A_1)| < 1$ and $|\sigma_{max}(A_4)| < 1$.

ii-->iii If $|\sigma_{max}(A_1)| < 1$ and $|\sigma_{max}(A_4)| < 1$ then there exists a k such that $||A_1{}^k|| < 1$, and $||A_4{}^k|| < 1$ (see [72]).■

Remark 2. For the special case $A_2 = A_3 = 0$ we showed that even though we started with a sufficient condition for stability we obtained the necessary and sufficient conditions for asymptotic stability. Thus we can discuss instability.

Lemma (9): For the case $A_2 = A_3 = 0$, the system is unstable if and only if

$$|\sigma_{max}(A_1)| > 1 \text{ or/and } |\sigma_{max}(A_4)| > 1$$

Proof: The proof can easily be obtained by the use of the previous theorem.∎

Similarly the following theorem can be stated for marginal stability.

Theorem (15'): The following statements are equivalent. For the case $A_2 = A_3 = 0$

i) The system is marginally stable

ii) $|\sigma_{max}(A_1)| \leq 1$ and $|\sigma_{max}(A_4)| \leq 1$

iii) $||A_1^k|| \leq 1$ and $||A_1^k|| \leq 1$ for some k.

Proof: Similar to the proof of the theorem 15.∎

Case 2 ($A_2 = 0$). In this case we have

$$\Phi(n;1) = \begin{bmatrix} (A_4)^T(A_4) & 0 \\ 0 & W(n;1) \end{bmatrix}$$

$$\Phi(n;2) = \begin{bmatrix} (A_4^2)^T(A_4^2) & 0 \\ 0 & W(n;2) \end{bmatrix}$$

$$\vdots$$

$$\Phi(n;i) = \begin{bmatrix} (A_4^i)^T(A_4^i) & 0 \\ 0 & W(n;i) \end{bmatrix}$$

The matrices $W(n;i)$ are not necessarily diagonal. Thus the eigenvalues of $\Phi(n;i)$ can not implicitly be determined in terms of eigenvalues of matrix A_1. However it can easily be seen that if $|\sigma_{max}(A_4)|$ or/and $|\sigma_{max}(A_1)|$ is/are greater than one then $\sigma_{max}[\Phi(n;i)] > 1$ for all i and n. This is due to the fact that $|\sigma_{max}(A_4)| > 1$ implies that $|\sigma_{max}(A_4^k)| > 1$ and thus $||A_4{}^k|| > 1$ for any k. On the other hand $|\sigma_{max}(A_1)| > 1$ implies $|\sigma_{max}[W(n;i)]| > 1$ (since $\sigma_{max}(A+B) > \sigma_{max}(A)$ if B is positive definite). Therefore the system is unstable. These facts can be summarized in the following lemma.

Lemma (10): For the case $A_2 = 0$, the system is unstable if

$$\sigma_{max}(A_1) > 1 \quad \text{or/and} \quad \sigma_{max}(A_4) > 1.$$

Proof: Similar to the proof of the lemma 9.■

A discussion similar to that for the case 2 can be made for the case $A_3 = 0$. In this case

$$\sigma_{max}\Phi(n;i) = \max\{\sigma[(A_1)^T(A_1)]U\sigma[V(n;i)]\}$$

where $V(n;i)$ is not necessarily diagonal and it is expressed in terms of A_1, A_2, and A_4.

4.6 LYAPUNOV STABILITY THEORY

To investigate the stability of a system of differential equations without having to solve them, Lyapunov devised his so-called second method in 1892. The idea involved is a generalization of the concept of energy for a conservative dynamic system. In such a system the energy is a positive function which decreases to zero as an equilibrium state is approached. If then, for a general system, a function with properties similar to those of an energy function can be found the stability of the system can be guaranteed. That is to say, if a function $V(x)$ can be constructed such that within some region around the critical point the contours of constant $V(x)$ represent concentric shells which decrease to zero as $||x||$ goes to zero and if the trajectories of the system cross these shells in an inward direction, then the critical point is asymptotically stable. Functions with the above properties are known as LYAPUNOV FUNCTIONS and we proceed now with formal definitions of these functions and with statements of the Lyapunov stability theorem.

Def (2): A function W: $R^n \longrightarrow R$ is positive (negative) definite, in a region δ containing the origin, if $W(0) = 0$, $W(x) > 0$ (<0) for all $x \neq 0$ in δ.

Def (2'): $V(x,k)$ is positive definite in δ if $V(0,k) = 0$, and $V(x,k) \geq W(x)$ for some positive definite $W(.)$ and all x in δ.

An immediate use of positivity is the following definition.

Def (3): $V(x,k)$ is a Lyapunov function for the system

$$x(k+1) = f(x(k),k), \quad f(0) = 0 \tag{27}$$

if:

i) $V(x,k)$ is continuous in x and $V(0,k) = 0$

ii) $V(x,k)$ is positive definite

iii) $\delta V(x(k),k) = V[f(x(k),k),k+1] - V[x(k),k]$ is negative definite.

The essence of the Lyapunov approach is captured in the following theorem.

Theorem (16) [73]: The solution $x(k) = 0$ is asymptotically stable if there exists a Lyapunov function for the system (27).

For the systems of interest in this study, the Lyapunov functions are essentially quadratic forms and the systems are linear. Hence every locally stable system is globally stable.

Given the difference equation

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{28}$$

it is possible to use the quadratic form

$$V(x,k) = x^T P(k)x \tag{29}$$

as a potential Lyapunov function. Forming

$$\delta V(x(k),k) = V(x(k+1),k+1) - V(x(k),k), \tag{30}$$

it is possible to prove

<u>Theorem (17)</u>: The origin of

$$x(k+1) = A(k)x(k)$$

is asymptotically stable if there exists a positive definite matrix P_0 such that the equation

$$A^T(k)P(k+1)A(k) - P(k) = -I, \quad P(0) = P_0 \tag{31}$$

has a bounded positive definite solution.

Now the question is how the standard 1-D Lyapunov theory can be extended to the 2-D setting. Two different methods have been explored. The first method [61] considers a 2-D Lyapunov equation with constant coefficients. The second method [63] considers a 1-D Lyapunov equation in which the functions are complex valued. In [60] a m-D continuous system is considered. It was [61] extended to m-D discrete systems by use of the bilinear transformation. They also stated some necessary and sufficient conditions for checking stability. It was shown [62] that even though the sufficiency part holds, the necessity part is not necessarily true.

Our approach considers the Wave model. It is known that the Wave model has the advantage of being 1-D. Therefore it is possible to apply the available theorems for 1-D systems to the Wave model. This will be done in the next section.

<u>Wave Model Case</u>

Consider the Wave model, with the assumption of $v(n) = f(n) = 0$,

$$\phi(n+1) = A(n)\phi(n)$$

Define the quadratic function

$$V(n) = \phi^T(n)P(n)\phi(n) \tag{32}$$

where $\alpha I < P(n)$ for some $\alpha > 0$;

and the incremental function $\delta V(n)$ by

$$\delta V(n) = V(n+1) - V(n). \tag{33}$$

Combining equations 32 and 33 yields

$$\delta V(n) = \phi^T(n+1)P(n+1)\phi(n+1) - \phi^T(n)P(n)\phi(n)$$

By substituting for $\phi(n+1)$, we obtain

$$\begin{aligned}\delta V(n) &= \phi^T(n)A^T(n)P(n+1)A(n)\phi(n) - \phi^T(n)P(n)\phi(n)\\ &= \phi^T(n)[A^T(n)P(n+1)A(n) - P(n)]\phi(n)\\ &= -\phi^T(n)Q(n)\phi(n)\end{aligned}$$

where

$$A^T(n)P(n+1)A(n) - P(n) = -Q(n). \tag{33'}$$

The model is clearly a time varying 1-D state equation. Unique to the Wave model is the fact that the transition matrix, $A(n)$, is not square.

We now state the following corollaries for determining the stability of the Wave model.

<u>Corollary (4)</u>: The equilibrium point, 0, at time n_0 of the W-A model, $\phi(n+1) = A(n)\phi(n)$, is asymptotically stable if there exists a positive definite $V(n)$ such that $-\delta V(n)$ is positive definite.

<u>Proof</u>: This proof follows from theorem 17.■

<u>Corollary (5)</u>: The origin of the system $\phi(n+1) = A(n)\phi(n)$ is asymptotically stable if there exists a positive definite matrix P_0 such that the solutions of

$$A^T(n)P(n+1)A(n) - P(n) = -I(n), \quad P(0) = P_0 \tag{34}$$

are positive definite and bounded.

In the following we restrict attention to the stationary Wave model and attempt to make use of the particular banded structure of the matrix A(n).

We recall that

$$A(n) = \begin{bmatrix} A_4 & 0 & 0 & 0 & 0 & 0 \cdots\cdots & 0 \\ A_2 & 0 & 0 & 0 & 0 & 0 \cdots\cdots & 0 \\ 0 & A_3 & A_4 & 0 & 0 & \cdots\cdots & 0 \\ 0 & A_1 & A_2 & 0 & \cdots & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\ & & & & & & A_3 \\ 0\ 0\ 0\ 0\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & A_1 \end{bmatrix}$$

Defining

$$\bar{A} = \begin{bmatrix} A_3 & A_4 \\ A_1 & A_2 \end{bmatrix}$$

$$J(n) = \begin{bmatrix} 0 \\ I(n) \\ 0 \end{bmatrix}$$

$D(n+1) = \text{diag}_{n+1}\ (\bar{A}),$

we see that $A(n) = D(n+1)J(n)$. Hence

$$A^T(n)P(n+1)A(n) = J^T(n)D^T(n+1)P(n+1)D(n+1)J(n).$$

If $\bar{A}$ is stable there exists a unique positive definite matrix P, as the solution of

$$\bar{A}^T P \bar{A} - P = -I.$$

Hence there exists a unique block diagonal $P(n+1) = \text{diag}\ (P)$ such that

$$D^T(n+1)P(n+1)D(n+1) = P(n+1) - I(n+1).$$

Thus

$$A^T(n)P(n+1)A(n) = J^T(n)P(n+1)J(n) - J^T(n)I(n+1)J(n)$$
$$= J^T(n)P(n+1)J(n) - I(n).$$

Defining $Q(n) = -A^T(n)P(n+1)A(n) + P(n) + I(n)$ we have

<u>Lemma (11)</u>: The Wave model is asymptotically stable if the following conditions are satisfied

i) $\bar{A}$ is stable

ii) $Q(n)$ is positive definite.

<u>Proof</u>: The result follows immediately from definition 3 .

Another sufficient condition can be obtained in terms of the eigenvalues of P.∎

<u>Lemma (12)</u>: The Wave model is asymptotically stable if the following conditions are satisfied

i) $\bar{A}$ is stable

ii) $\sigma_{max}(P) < 1 + \sigma_{min}(P)$

Proof: Define

$$V(n+1) = \langle \phi(n+1), P(n+1)\phi(n+1)\rangle$$
$$= \langle J(n)\phi(n), D^*(n+1)P(n+1)D(n+1)J(n)\phi(n)\rangle$$
$$= \langle J(n)\phi(n), [P(n+1)-I(n+1)]J(n)\phi(n)\rangle$$
$$\leq (\sigma_{max}(P)-1)\ \langle J(n)\phi(n), J(n)\phi(n)\rangle$$
$$\leq (\sigma_{max}(P)-1)\ \langle \phi(n), \phi(n)\rangle$$

On the other hand

$$\langle \phi(n), P(n)\phi(n)\rangle \geq \sigma_{min}(P)\ \langle \phi(n), \phi(n)\rangle.$$

Thus

$$V(n+1) \leq \{[\sigma_{max}(P) - 1]/\sigma_{min}(P)\}V(n)$$

by condition ii) $V(n+1) < V(n)$ and $V(n)$ is a Lyapunov function for the Wave model.■

The previous result suggests the following alternative: Choose the matrix $P = \mathrm{diag}\,(P_4\ \ P_1)$

$$P(n) = \begin{bmatrix} P_4 & 0 & 0 & \cdots\cdots & 0 \\ 0 & P_1 & 0 & \cdots\cdots & 0 \\ \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & P_4 & 0 \\ 0 & 0 & & & P_1 \end{bmatrix} = \mathrm{diag}_n\,(P)$$

Solving equation 34 yields the following set of equations:

$$1)\quad A_3^T P_4 A_3 + A_1^T P_1 A_1 - P_1 = -Q_1 \qquad (35.1)$$

$$2)\quad A_4^T P_4 A_4 + A_2^T P_1 A_2 - P_4 = -Q_4 \qquad (35.2)$$

$$3)\quad A_3^T P_4 A_4 + A_1^T P_1 A_2 = -Q_2 \qquad (35.3)$$

These equations can be written in the following compact form

$$A^T P A - P = -Q$$

where A is the state matrix of G-R model and P, Q are as defined earlier in this section.

We can now state the following theorem.

Theorem (18): The origin of system of equation 2 is asymptotically stable if there exists a symmetric positive definite matrix $P = \text{diag}(P_1 \quad P_4)$ such that Q is positive definite.

Proof: Similar to the proof of corollary 5.■

The above conditions may seem very restrictive; however the analysis of the scalar case offers interesting insight into the structure of systems satisfying it. Consider the scalar case

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_4 \end{bmatrix} \qquad P = \begin{bmatrix} p_1 & 0 \\ 0 & p_4 \end{bmatrix}$$

Assume $q_1 > 0$ and $q_4 > 0$. The equations 35.1 and 35.2 will yield a unique positive definite solution P if the following conditions are satisfied:

1) $\gamma = (1-a^2{}_1)(1-a^2{}_4) - (a_2 a_3)^2 > 0$

2) $|a_1| < 1$

3) $|a_4| < 1$

Now we would like to obtain a positive definite matrix Q for every $q_1 > 0$ and $q_4 > 0$ such that equation 35.3 holds. This can be done if the following conditions are satisfied

4) $\gamma^2 + 2(a_2 a_3)^2[a^2{}_4(1-a^2{}_1)+a^2{}_1(1-a^2{}_4)] + 2a_1 a_2 a_3 a_4[(1-a^2{}_1)(1-a^2{}_4)+(a_2 a_3)^2] > 0$

5) $a^4{}_2a^2{}_3a^2{}_4 + a^2{}_1a^4{}_2(1-a^2{}_4)^2 + 2a_1a^3{}_2a_3a_4(1-a^2{}_4) > 0$

6) $a^2{}_1a^2{}_2a^4{}_3 + a^4{}_3a^2{}_4(1-a^2{}_1)^2 + 2a_1a^3{}_3a_2a_4(1-a^2{}_1) > 0$

Thus conditions 1-6 are sufficient for existence of a desired solution.

It is worthwhile to point out that if $a_1a_2a_3a_4 > 0$ then conditions 4-6 are always satisfied. Thus, for this case, only conditions 1-3 are required. It is also interesting to note that as a_2 and a_3 increase, a_1 and a_4 should approach zero if condition 1 must remain valid. However if $a_2a_3 = 0$, then stability of a_1 and a_4 are sufficient.

We now examine the following general case. Given that Q is positive definite, under what conditions does there exist a unique solution for P? Our approach uses Kronecker theory [74]. The system equation 35 can be written in the following form

$$\begin{bmatrix} A_1^T \otimes A_1^T - I & A_3^T \otimes A_3^T \\ A_2^T \otimes A_2^T & A_4^T \otimes A_4^T - I \\ A_2^T \otimes A_1^T & A_4^T \otimes A_3^T \end{bmatrix} \begin{bmatrix} \underline{P}_1 \\ \underline{P}_4 \end{bmatrix} = \begin{bmatrix} -\underline{Q}_1 \\ -\underline{Q}_4 \\ -\underline{Q}_2 \end{bmatrix} \tag{36}$$

where $\underline{P} = [\underline{P}_1 \quad \underline{P}_4]^T$, $\underline{Q} = [\underline{Q}_1 \quad \underline{Q}_4 \quad \underline{Q}_2]^T$ are (n^2+m^2) and (n^2+m^2+nm) vectors formed by aggregating the columns of P and Q. We will simplify the notation into the more compact form $\underline{A}\underline{P} = -\underline{Q}$.

It is well known result that if Rank[$\underline{A}$] = Rank[$\underline{A}$:$\underline{Q}$] then there exists a solution for above equation. Furthermore if the Rank[$\underline{A}$] is equal to (n^2+m^2) then it has a unique solution.

We now use a different approach. This approach takes advantage of determinant theory. Since the matrix A is not a square matrix we can not discuss the uniqueness of the solution of equation 36 by checking the determinant of A. However considering equations 35.1 and 35.2 we can write the following

$$\begin{bmatrix} A_1^T \otimes A_1^T - I & A_3^T \otimes A_3^T \\ A_2^T \otimes A_2^T & A_4^T \otimes A_4^T - I \end{bmatrix} \begin{bmatrix} \underline{P}_1 \\ \underline{P}_4 \end{bmatrix} = \begin{bmatrix} -\underline{Q}_1 \\ -\underline{Q}_4 \end{bmatrix} \qquad (37)$$

or more simply $\underline{\hat{A}}\underline{P} = -\underline{\hat{Q}}$.

Now the following questions are in order. First given Q_1 and Q_4 positive definite, under what condition does there exist a unique solution for P_1 and P_4? Second, assuming the obtained matrix P is positive definite, then under what conditions the matrix Q_2 obtained from equation 3 by substituting for P_1 and P_4 make the matrix Q positive definite.

The equation 37 has a unique solution if $\det(\underline{\hat{A}}) \neq 0$. It can easily be verified that

$$\begin{aligned} \det[\underline{\hat{A}}] &= \det[A_1^T \otimes A_1^T - I] \,.\, \det[A_4^T \otimes A_4^T - I - A_3^T \otimes A_3^T \\ &\qquad (A_1^T \otimes A_1^T -)^{-1}(A_2^T \otimes A_2^T)] \\ &= \det[A_4^T \otimes A_4^T - I] \,.\, \det[A_1^T \otimes A_1^T - I - A_2^T \otimes A_2^T \\ &\qquad (A_4^T \otimes A_4^T - I)^{-1}(A_3^T \otimes A_3^T)]. \end{aligned}$$

$\det[A_1^T \otimes A_1^T - I] \neq 0$ and $\det[A_4^T \otimes A_4^T - I] \neq 0$ are necessary conditions for existence of a unique positive solution P. It can be easily verified that $\det[A_1^T \otimes A_1^T -$

$I] \neq 0$ (respectively $\det[A_4^T \otimes A_4^T - I] \neq 0$) if and only if $1 - |\sigma_i||\sigma_j| \neq 0$ for all i and j, where σ_i are eigenvalues of A_1 (respectively A_4). This means in order to have a unique positive definite solution P it is necessary that A_1 and A_4 not have a unity eigenvalue. Note that $|\sigma_i| < 1$ is a sufficient condition for having $\det[A_1^T \otimes A_1^T - I] \neq 0$ (respectively $\det[A_4^T \otimes A_4^T - I] \neq 0$). It should be noted that the matrix A can have a unity eigenvalue and still have a unique solution P. This is because matrix P is diagonal. Moreover in [63] there are sufficient conditions for existence of positive definite P.

Now assume there exists a unique positive definite P such that equations 35.1 and 35.2 are satisfied. We now try to answer the second question. From equation 37, we have

$$\begin{bmatrix} \underline{P}_1 \\ \underline{P}_2 \end{bmatrix} = \begin{bmatrix} A_1^T \otimes A_1^T - I & A_3^T \otimes A_3^T \\ A_2^T \otimes A_2^T & A_4^T \otimes A_4^T - I \end{bmatrix}^{-1} \begin{bmatrix} -\underline{Q}_1 \\ -\underline{Q}_4 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} -\underline{Q}_1 \\ -\underline{Q}_4 \end{bmatrix}$$

$E = [A_1^T \otimes A_1^T - I]^{-1}\{I + [A_3^T \otimes A_3^T]H[A_2^T \otimes A_2^T][A_1^T \otimes A_1^T - I]^{-1}\}$

$F = -[A_1^T \otimes A_1^T - I]^{-1}[A_3^T \otimes A_3^T]H$

$G = -H[A_2^T \otimes A_2^T][A_1^T \otimes A_1^T - I]^{-1}$

$H = \{[A_4^T \otimes A_4^T - I] - [A_2^T \otimes A_2^T][A_1^T \otimes A_1^T - I]^{-1}[A_3^T \otimes A_3^T]\}^{-1}$

Equation 35.3 can be written in the following form

$Q_2 = -[A_1^T \quad A_3^T]\ [P_1 \oplus P_4]\ [A_2^T \quad A_4^T]^T$

$$= -[A_1^T \quad A_3^T]\,[(-EQ_1-FQ_4) \oplus (-GQ_1-HQ_4)]\,[A_2^T \quad A_4^T]^T$$

by substituting for Q_2 in Q. Then Q is in terms of Q_1 and Q_4. Thus positive definite matrices Q_1 and Q_4 should be chosen such that the matrix Q is positive definite.

Since checking the above condition is tedious we try to find only sufficient conditions for the existence of a positive definite Q. For this let $x = [x_1 \; x_2]^T$ with $x_1 \in R^n$, $x_2 \in R^m$ then,

$$\begin{aligned} x^TQx &= x_1^TQ_1x_1 + x_2^TQ_4x_2 + 2x_1^TQ_2x_2 \\ &\geq x^T[(Q_1 \oplus Q_4) - ||Q_2||I]x \qquad (39) \end{aligned}$$

This indicates that the matrix Q will be positive definite if $[(Q_1 \oplus Q_4) - ||Q_2||I]$ is positive definite. By choosing $Q_1 = I$ and $Q_4 = I$ equation 39 will be

$$x^TQx \geq x^T(1-||Q_2||)x.$$

Thus Q is positive definite if $||Q_2|| < 1$.

4.7 EXAMPLES

In this section we present several examples which serve to demonstrate the several stability criteria of earlier sections.

In our first example we illustrate theorem 10, which asserts that, for stationary coefficients, the G-R model is stable provided $\sigma_{max}(A^TA) < 1$.

Example (1): Given the system of equation 2, with the following matrices,

$A_1 = [0.5]$ $\qquad\qquad$ $A_2 = [0.5 \quad 0]$

$$A_3 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix} \qquad\qquad A_4 = \begin{bmatrix} -0.5 & 0 \\ 0.3 & 0.1 \end{bmatrix}$$

it follows easily that $Q = A^T A$ is given by

$$Q = \begin{bmatrix} 0.66 & 0.12 & 0.04 \\ 0.12 & 0.59 & 0.03 \\ 0.04 & 0.03 & 0.01 \end{bmatrix}.$$

From corollary 1 we have $\delta = \max\{0.82, \; 0.74, \; 0.08\} = 0.82$. Thus $\sigma_{max}(Q) \leq 0.82 < 1$. Therefore by theorem 10 the system is asymptotically stable. We can arrive at the same result by a different approach. First of all from the G-R model it is clear that the system consists of a 2-D system and a 1-D system. The 2-D system has the following state matrix

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$

and the 1-D system has state matrix $A = [0.1]$. Obviously the 1-D system is stable. The 2-D system is also shown to be stable by Lu-Lee [65].

<u>Example (2)</u>: Given the system of equation 2 with the following matrices

$$A_1 = [0.5] \qquad\qquad A_2 = [0.5 \quad 0]$$

$$A_3 = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix} \qquad A_4 = \begin{bmatrix} -0.5 & 0 \\ 0.3 & 0.1 \end{bmatrix}$$

Determine if the system is stable.

Solution: The matrix $Q = A^T A$ is given by

$$Q = \begin{bmatrix} 0.99 & 0.21 & 0.07 \\ 0.21 & 0.59 & 0.03 \\ 0.07 & 0.03 & 0.01 \end{bmatrix}.$$

From corollary 1 $\delta = \max\,[1.27,\ 0.83,\ 0.11] = 1.27$. Since $|1.27| > 1$, the 1-step test fails. Furthermore $\sigma_{max}(Q) = 1.09 > 1$ Therefore theorem 10 can not also be used. Thus we can not make any conclusion regarding stability. However, this system is asymptotically stable. It has the same characteristic polynomial as example 1.

Example (3): Given the matrix

$$A = \begin{bmatrix} 1/3 & (\sqrt{3})2^{(m-4)/2} \\ 0 & \sqrt{1/2} \end{bmatrix}$$

we wish to check the stability of the system for different values of m. The relevant calculations of $\Phi(n:i)$ are the following.

Case 1: $m = 1$ then

$$\Phi(1:1) = \begin{bmatrix} 7/8 & 0 \\ 0 & 1/9 \end{bmatrix}$$

$$\Phi(2:1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.9 \end{bmatrix}.$$

Since $\sigma_{max}[\Phi(2:1)] = 0.95 < 1$, the 1-step test is conclusive. Thus the system is asymptotically stable.

<u>Case 2</u>: $m = 2$

$$\Phi(1:1) = \begin{bmatrix} 5/4 & 0 \\ 0 & 1/9 \end{bmatrix}$$

$$\Phi(1:2) = \begin{bmatrix} 17/24 & 0 \\ 0 & 1/81 \end{bmatrix}$$

$\sigma_{max}[\Phi(1:1)] > 1$, which indicates that the 1-step test is inconclusive. However $\sigma_{max}[\Phi(1:2)] < 1$, thus the 2-step test is conclusive. Thus the system is asymptotically stable.

<u>Case 3:</u> $m = 3$

$$\Phi(1:1) = \begin{bmatrix} 2 & 0 \\ 0 & 1/9 \end{bmatrix}$$

$$\Phi(1:2) = \begin{bmatrix} 7/6 & 0 \\ 0 & 1/81 \end{bmatrix}$$

$$\Phi(1:3) = \begin{bmatrix} 65/108 & 0 \\ 0 & (1/3)^6 \end{bmatrix}$$

$\sigma_{max}[\Phi(1:1)] = 2 > 1$, $\sigma_{max}[\Phi(1:2)] = 7/6 > 1$, $\sigma_{max}[\Phi(1:3)] = 65/108 < 1$. Therefore the system is asymptotically stable. Note that this example is constructed such that for $m = i$ the first $(i-1)$-step tests are inconclusive and i-step test is conclusive.

Our next example suggests that the nonstationary test of theorem 9 may be useful for even stationary m-D systems.

Example (4): Given the difference equation

$$\phi(i+1,j+1) = ab\phi(i,j) + bu(i,j+1) + au(i+1,j)$$

it follows easily that $\Phi(z_1,z_2)/U(z_1,z_2) = H(z_1,z_2)$ is of the form

$$H(z_1,z_2) = \frac{az_1 + bz_2}{z_1z_2 - ab} .$$

The minimal local state realization of $H(z_1,z_2)$ is nonstationary (see [75] for details). Indeed for

$$\mu(i+1,j) = -a(-1)^{i+j}r(i,j) + bu(i,j)$$

$$r(i,j+1) = b(-1)^{i+j}\mu(i,j) + au(i,j)$$

it follows that

$$\phi(i,j) = \mu(i,j) + r(i,j).$$

On the other hand the stationary state realization has 3 variables and the following matrices

$$A = \begin{bmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix}$$

$$C = [b/a \quad 0 \quad a/c]$$

<u>Stability Analysis</u> (assuming $|a| < 1$, $|b| < 1$)

<u>Case 1:</u> (nonstationary)

$Q(i,j) = A^T(i,j)A(i,j)$

$$= \begin{bmatrix} 0 & b(-1)^{i+j} \\ -a(-1)^{i+j} & 0 \end{bmatrix} \begin{bmatrix} 0 & -a(-1)^{i+j} \\ b(-1)^{i+j} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b^2 & 0 \\ 0 & a^2 \end{bmatrix}.$$

The above assumption implies that the system is asymptotically stable. This is due to theorem 10 since $Q(i,j) = Q$ for all i, j and $\sigma_{max}[Q(i,j)] = \max\{b^2, a^2\} < 1$.

<u>Case 2:</u> (stationary)

To apply the 1-step test we need to calculate

$$Q = A^TA = \begin{bmatrix} b^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However $\sigma_{max}(Q)$ depends also on the value of c. For instance if $(a^2+c^2) > 1$ then $\sigma_{max}(Q) > 1$. Therefore the 1-step test fails.

When using 2-step test we should calculate the matrix

$\Phi(n:2) = [A(n+1)A(n)]^T[A(n+1)A(n)]$. It can easily be seen that

$$G = \begin{bmatrix} (ab)^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad F = b^2(a^2+c^2), \quad H = L = S = R = P = 0.$$

Thus $\sigma_{max}[\Phi(n:2)] = \max\{(ab)^2, \ b^2(a^2+c^2)\} = b^2(a^2+c^2)\}$ for all n. If $b^2(a^2+c^2) < 1$ we can conclude that the system is asymptotically stable. However if $b^2(a^2+c^2) > 1$ the 2-step test fails. In this case we must try the 3-step test.

Similarly we need to find the $\sigma_{max}[\Phi(n:3)]$. After some manipulation we obtain

$$\sigma_{max}[\Phi(n:3)] = (ab)^2(a^2 + c^2) \quad \text{for all n.}$$

$(ab)^2(a^2 + c^2) < 1$ implies that the system is asymptotically stable. But if this value is greater than or equal to unity we need to go to the next step.

Note that not only the nonstationary state model has smaller dimension than the stationary state model but also that the stability analysis can be performed more easily.

In contrast, applying Jury's table test to above transfer function implies that the system is BIBO iff $|ab| < 1$. Clearly $|a| < 1$, $|b| < 1$ implies that $|ab| < 1$. However by assuming $|ab| < 1$ we obtain the same conclusion but by checking a greater number of steps. In this case, considering the nonstationary model the 1-step test fails. However $\Phi(n:2) = \text{diag}[a^2b^2]$ since $T = F = a^2b^2$ and $H = L = S = R = P = 0$. Thus by assumption $\sigma_{max}[\Phi(n:2)] = a^2b^2 < 1$.

Therefore the system is asymptotically stable. Using the stationary model the same conclusion can be made in a greater number of steps.

Example (5): Given a nonstationary 2-D system in the form of the G-R model with the state matrix

$$A(i,k) = \begin{bmatrix} a\cos(i) & a\sin(i) \\ -b\sin(i) & b\cos(i) \end{bmatrix},$$

then

$$Q(i,k) = A^T(i,k)A(i,k) = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}.$$

Note that $Q(i,k)$ is independant of i and k. Thus if $a^2 < 1$ and $b^2 < 1$ then $\sigma_{max}[Q(i,k)] < 1$. This implies that the product of $Q(i,k)$ goes to zero as i and k go to ∞. By theorem 10 the system is asymptotically stable.

Considering the case $a = b = 1$ yields that A is a unitary matrix (i.e. $A^T(i,k)A(i,k) = I$). Thus the system is marginally stable. Note that $\sigma(A) = e^{+ji}$ which implies that $|\sigma_{max}[A(i,k)]| = 1$. It also implies that the best we can hope for is marginal stability.

Assuming the matrix A is in terms of both variables

$$A(i,k) = \begin{bmatrix} \cos(i) & \sin(i) \\ -\sin(k) & \cos(k) \end{bmatrix}$$

then

$$Q(i,k) = A^T(i,k)A(i,k) = \begin{bmatrix} 1 & \sin(i-k) \\ \sin(i-k) & 1 \end{bmatrix}.$$

The maximum eigenvalue of $Q(i,k)$ is either $(1+\sin(i-k))$ or $(1-\sin(i-k))$. That is to say that $\sigma_{max}[Q(i,k)]$ depends upon the sign of $\sin(i-k)$ and is possibly greater than unity. However

$$\prod_{i,k} [1+\sin(i-k)][1-\sin(i-k)] = \prod_{i,k} \cos^2(i-k) \le 1$$

Thus the system is marginally stable.

Example (6): Given the system of equation 2 with the following matrices

$$A_1 = [0.5] \qquad A_2 = [0.5 \quad 0]$$

$$A_3 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.1 \end{bmatrix}$$

Determine if the system is stable.

Solution: First calculate $Q = A^TA$, namely

$$Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}.$$

Since $\sigma_{max}(Q) = 0.5 < 1$, by theorem 10, the system is asymptotically stable. Moreover since the matrix is symmetric we need only to check the maximum eigenvalue of the matrix A as a 1-D problem. $|\sigma_{max}(A)| = .5 < 1$. Thus by corollary 3 the system is asymptotically stable.

Example (7): Given the system matrix

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix},$$

by the use of Lyapunov theory determine if the system presented by the Wave model is asymptotically stable.

Solution: Let's consider the second case.

From 35.1 and 35.2 the following equations need to be solved

$$(0.5)P_4(0.5) + (0.5)P_1(0.5) - P_1 = -1$$

$$(-0.5)P_4(-0.5) + (0.5)P_1(0.5) - P_4 = -1$$

The solution is $P_1 = 2$, $P_4 = 2$.

It can easily be seen that equation 35.3 is also satisfied:

$$(0.5)(2)(-0.5) + (0.5)(2)(0.5) = 0.$$

Since the matrix

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is positive definite the system of equation 2 is asymptotically stable. Note that this example is taken from [65] and has been shown to be stable.

Example (8): Given the system matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Using Lyapunov theory, what is the condition on a and b in order for the system to be asymptoticaly stable?

Solution: Equation 35 implies the following

$$1) \qquad a p_1 a - p_1 = -1$$

$$2) \qquad b p_4 b - p_4 = -1.$$

The solution for p_1 and p_4 is $p_1 = 1/(1-a^2)$, $p_4 = 1/(1-b^2)$. For stability, p_1 and p_4 must be positive definite. Thus

$$1/(1-a^2) > 0, \quad 1/(1-b^2) > 0.$$

This implies that $a^2 < 1$, and $b^2 < 1$ (that is $|a| < 1$ and $|b| < 1$). Clearly this result is the same as the result given in [65].

Example (9): Given the system matrix

$$A(i,j) = \begin{bmatrix} 0 & -a(-1)^{i+j} \\ b(-1)^{i+j} & 0 \end{bmatrix}.$$

Using Lyapunov theory what is the condition on a and b in order for the system to be asymptoticaly stable?

Solution: By considering the second case we have

$$b(-1)^{i+j} p_4 b(-1)^{i+j} - p_1 = -1$$

$$(-a)(-1)^{i+j} p_1 (-a)(-1)^{i+j} - p_4 = -1$$

By solving above equations we obtain

$$p_1 = (1+a^2b^2)/(1-a^2b^2)$$

$$p_4 = 2/(1-a^2b^2).$$

For asymptotic stability p_1 and p_4 should be positive definite. That is, $1-a^2b^2 > 0$. This implies $|ab| < 1$. This is the same result as was obtained in example 5.

Example (10): Given the system matrix

$$A(i,j) = \begin{bmatrix} a\cos(i) & a\sin(i) \\ -b\sin(i) & b\cos(i) \end{bmatrix}.$$

Using Lyapunov theory what is the condition on a and b in order for the system to be asymptotically stable?

Solution: Equation 35 implies the following

$$b^2 \sin^2(i)\, p_4 + a^2 \cos^2(i)\, p_1 - p_1 = -1$$

$$b^2 \cos^2(i)\, p_4 + a^2 \sin^2(i)\, p_1 - p_4 = -1$$

$$-b^2 \sin(i)\cos(i)\, p_4 + a^2 \cos(i)\sin(i)\, p_1 = 0.$$

Solving the last equation yields that $a^2p_1 = b^2p_4$. After substituting for a^2p_1 and b^2p_4 in the first equations we obtain

$$a^2 \sin^2(i)\, p_1 + a^2 \cos^2(i)\, p_1 - p_1 = -1$$

$$b^2 \cos^2(i)\, p_4 + b^2 \sin^2(i)\, p_4 - p_4 = -1.$$

Then $p_1 = 1/(1-a^2)$ and $p_4 = 1/(1-b^2)$. For asymptotic stability p_1 and p_4 should be positive definite. That is $1-a^2 > 0$ and $1-b^2 > 0$. This implies $|a| < 1$ and $|b| < 1$. This is the same result as we obtained in example 6.

CHAPTER 5

STABILIZATION OF THE WAVE MODEL

ABSTRACT

This chapter considers the use of feedback to stabilize m-D systems represented by a nonstationary Wave model. Several types of feedback are possible. As is the case with 1-D systems, it is possible to consider either output or state information. Unique to the m-D case is the possibility of using interior or boundary inputs in the feedback channel. The developement presents the feedback problem in a format that is compatible with the stability definitions of chapter 4. Numerical examples are given to illustrate the various techniques.

5.1 INTRODUCTION

Consider now the Wave model

$$\phi(n+1) = A(n)\phi(n) + B(n)v(n) + E(n)f(n)$$

$$\mu(n) = C(n)\phi(n) + D(n)v(n) + H(n)f(n). \qquad (1)$$

presented in section 4.1. Recall that $v(.)$ and $f(.)$ represent interior and boundary controls respectively. The notation $\phi(.)$ and $\mu(.)$ denote the state and output responses.

A general form of feedback is given by the following equations.

$$v(n) = K(n)\phi(n) + R(n)\mu(n) + u(n)$$

$$f(n) = L(n)\phi(n) + S(n)\mu(n) + g(n) \qquad (2)$$

Here K(.), R(.), L(.), S(.) represent the indicated state/output feedback to the interior/boundary controls, respectively. Since state stability is the issue of interest here, it suffices to consider u(.) = 0 and g(.) = 0. The closed loop state response is governed then by the equation

$$\phi(n+1) = [A(n) + B(n)K(n) + E(n)L(n) + B(n)R(n)C(n) + E(n)S(n)C(n)]\phi(n)$$

$$= T(n)\phi(n). \qquad (3)$$

It is apparent that T(.) inherits some of the characteristics of A(.), namely the growth in dimension. It is not true, however, that T(n) will automatically have the banded structure of equation 1. Indeed the quarter plane causality (synonymous with the banded structure) of the original system can be lost through even memoryless state and/or output feedback.

In the present chapter we consider the choice of K(.), R(.), L(.), S(.) such that T(n) has the banded structure and satisfies the stability criteria of chapter 4. Concerning the banded structure straightforward manipulations reveal that K(n), R(n), L(n), S(n) must have the following form;

$$K(n) = \begin{bmatrix} K_2(n,0) & 0 & 0 & 0 \cdots\cdots 0 & & 0 \\ 0 & K_1(n-1,1) & K_2(n-1,1) & 0 \cdots\cdots 0 & & 0 \\ 0 & 0 & 0 & 0 \cdots\cdots 0 & & 0 \\ \cdot & \cdot & \cdot\ \cdot & \cdot & & \\ & & & K_1(1,n-1) & K_1(1,n-1) & 0 \\ 0 & 0 & 0 & 0 \cdots\cdots 0 & & K_1(0,n) \end{bmatrix} \tag{4}$$

$$R(n) = \begin{bmatrix} R(n,0) & 0 & 0 \cdots\cdots & 0 \\ 0 & R(n-1,1) & 0 \cdots\cdots & 0 \\ 0 & 0 & R(n-2,2) & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \cdots R(0,n) \end{bmatrix} \tag{5}$$

$$L(n) = \begin{bmatrix} L_1(n,0) & 0 & 0 \cdots\cdots 0 \\ 0 & 0 \cdots\cdots 0 & .L_2(0,n) \end{bmatrix} \tag{6}$$

$$S(n) = \begin{bmatrix} S_1(n,0) & 0 & 0 \cdots\cdots 0 \\ 0 & 0 \cdots\cdots & S_2(0,n) \end{bmatrix}. \tag{7}$$

The dimension of these matrices is

$K(n)$ - $(n+1) \times (2n)$ $\qquad$ $L(n)$ - $(2) \times (2n)$

$R(n)$ - $(n+1) \times (n+1)$ $\qquad$ $S(n)$ - $(2) \times (n+1)$.

With these constraints the matrix $T(n)$ will be of the form

$$T(n) = \begin{bmatrix} \bar{A}_4(n,0) & 0 & 0 & 0 & 0 \\ \bar{A}_2(n,0) & 0 & 0 & 0 & 0 \\ 0 & \bar{A}_3(n-1,1) & \bar{A}_4(n-1,1) & 0 & 0 \\ 0 & \bar{A}_1(n-1,1) & \bar{A}_2(n-1,1) & 0 & \\ 0 & & & & \\ \vdots & \cdot & & & \\ \vdots & & \cdot & & \\ \vdots & & & \cdot & \\ \vdots & & & & \cdot \\ \vdots & & & & \\ 0 & 0 & 0 & & \bar{A}_3(0,n) \\ 0 & 0 & 0 & & \bar{A}_1(0,n) \end{bmatrix} \qquad (8)$$

where

$$\bar{A}_1(k,l) = [A_1 + B_2K_1 + B_2RC_1 + A_2L_2 + A_2S_2C_1](k,l)$$

$$\bar{A}_2(k,l) = [A_2 + B_2K_2 + B_2RC_2 + A_1L_1 + A_1S_1C_2](k,l)$$

$$\bar{A}_3(k,l) = [A_3 + B_1K_1 + B_1RC_1 + A_4L_2 + A_4S_2C_1](k,l)$$

$$\bar{A}_4(k,l) = [A_4 + B_1K_2 + B_1RC_2 + A_3L_1 + A_3S_1C_2](k,l).$$

We consider first the application of the stability criteria in theorems 4.1 and 4.2. For this purpose it is required to investigate the eigenvalues of the matrix $\bar{\Phi}(n) = T^T(n)T(n)$. It can easily be verified that $\bar{\Phi}(n)$ is square (2n x 2n), symmetric, block diagonal, and has the following form

$$\bar{\Phi}(n) = \begin{bmatrix} \bar{M}(n,0) & 0 & 0 & 0 \cdots\cdots 0 & \\ 0 & \bar{Q}(n-1,1) & 0 & 0 \cdots\cdots 0 & \\ 0 & & \bar{Q}(n-2,2) & & \\ 0 & & \cdot & & \\ \cdot & & & \cdot & \\ 0 & & & & \\ 0 & & & \bar{Q}(1,n-1) & 0 \\ 0 \cdots\cdots & \cdots\cdots 0 & & & \bar{N}(0,n) \end{bmatrix}, \qquad (9)$$

where the matrix Q has the form

$$\bar{Q}(k,l) = \begin{bmatrix} \bar{M}(k,l) & \bar{P}(k,l) \\ \bar{P}^T(k,l) & \bar{N}(k,l) \end{bmatrix} \tag{9'}$$

with

$$\bar{M}(k,l) = \bar{A}_4^T(k,l)\bar{A}_4(k,l)+\bar{A}_2^T(k,l)\bar{A}_2(k,l)$$

$$\bar{N}(k,l) = \bar{A}_3^T(k,l)\bar{A}_3(k,l)+\bar{A}_1^T(k,l)\bar{A}_1(k,l)$$

$$\bar{P}(k,l) = \bar{A}_3^T(k,l)\bar{A}_4(k,l)+\bar{A}_1^T(k,l)\bar{A}_2(k,l).$$

A useful additional property of $\Phi(n)$ is identified in the next lemma.

Lemma (1): If K(n), R(n), L(n), and S(n) conform to the requirements of equations 4, 5, 6, and 7 respectively, then $\Phi(n) = T^T(n)T(n)$ is block diagonal and

$$||T(n)||^2 = \sigma_{max}\bar{\Phi}(n) = \max\ \{\sigma[\bar{Q}(n-i,i)]\ ;\ i=0,1,\ldots n\}$$

where the $Q(i,j)$ are specified in equation 9'.

Proof: In equation 9 we see that $\bar{\Phi}(n)$ is block diagonal. We note also that $\bar{M}(n,0)$, $\bar{N}(0,n)$ are principal minors of $\bar{Q}(n,0)$, $\bar{Q}(0,n)$ respectively.■

The above development places in perspective the problem of designing feedback to achieve stability. One might consider each of the feedback types, represented by K(.), R(.), L(.), and S(.), individually or collectively. Lemma 1, then, translates the 1-step transition norm criteria to the feedback setting. It is also apparent that the other stability criteria of chapter 4 (m-step, Lyapunov, symmetric, stationary, etc) have analogous translates for closed-loop systems.

In the present chapter we shall explore a few of the several special cases. We demonstrate the feasibility of feedback design for stabilization, identify the limitations and difficulties and conclude with examples which illustrate the techniques involved.

5.2 STATE FEEDBACK STABILIZATION

In this section we consider state feedback to interior controls. In short R(.), L(.), and S(.) are identically zero while K(.) is arbitrary. The homogeneous transition equation then takes the form

$$\begin{aligned}\phi(n+1) &= [A(n) + B(n)K(n)]\phi(n)\\ &= T_1(n)\phi(n) \qquad (10)\end{aligned}$$

5.2.1 Stability Analysis

From 1-D state feedback theory it is possible to stabilize the system provided there exist a matrix K such that the closed-loop system is asymptotically stable. Note that the size of $T_1(n)$ increases as n increases and moreover that $T_1(n)$ is not a square matrix.

1-Step

Def (1): The system of equation 1 is said to be one-step state feedback stabilizable if there exists a matrix K(n) such that the system of equation 10 is asymptotically stable [see def 4.1].

Now the following corollaries are immediate.

Corollary (1): The system of equation 10 is aymptotically stable if

$$\prod_{i=k}^{p} ||A(i) + B(i)K(i)||$$

goes to zero as p goes to ∞.

Proof: The proof of this corollary follows directly from theorem 4.1. This is due to the fact that A(n) and [A(n)+B(n)K(n)] have an identical structure.■

Corollary (2): The system of equation 10 is aymptotically stable if $||A(i) + B(i)K(i)|| < 1-\varepsilon$ for i= 1,2,3 ... where $\varepsilon > 0$.

Proof: Similar to the proof of above corollary but using the theorem 4.2.■

By the use of lemma 1 the asymptotic stability can be determined by theorems 4.8 and 4.9. However, it should be noted that the equation 4.17, 4.18, and 4.19 must be modified respectively as follows:

(i) $\bar{\sigma}_Q(n) = \max_j \{\sigma_{max}{}^{1/2}\bar{Q}(n-j,j) \ : \ j=0,1,..,n\}.$

(ii) $\bar{\sigma}_Q(n) = \max_j \{||[A+BK](n-j,j)|| \ : \ j=0,1,..,n\}.$

(iii) $\bar{\sigma}_Q(n) = \max_j \{|\sigma_{max}[A+BK](n-j,j)| \ : \ j=0,1,..,n\}.$

Considering the stationary case, theorem 4.10 should be used in order to determine the asymptotic stability. For the special case of symmetric A, corollary 4.3 determines the asymptotic stability.

It is interesting to note that the conditions for stabilization are stated in terms of the state matrix A of the G-R model. Therefore considering the G-R model, it can be said that a 2-D system is stabilizable by using the state feedback if there exists a matrix $K(i,j)$ such that

$$||[A+BK](i,j)|| < 1 \qquad \text{for all } i \text{ and } j. \tag{11}$$

For the stationary case condition, 11 is simplified to

$$||A+BK|| < 1. \tag{12}$$

Furthermore if $[A+BK]$ is symmetric, then condition 12 turns to

$$|\sigma_{max}[A+BK]| < 1. \tag{13}$$

2-Step

Up to now all the theorems which have been stated are in the sense of definition 1. If $||A(n) + B(n)K(n)|| \geq 1$, then all above theorems are inconclusive. However, other definitions can be considered. Substituting for $\phi(n)$ in terms of $\phi(n-1)$ in equation 10 yields

$$\phi(n+1) = [A(n) + B(n)K(n)][A(n-1) + B(n-1)K(n-1)]\phi(n-1). \tag{14}$$

Now the following definition is in order.

Def (2): The system of equation 1 is said to be 2-step state feedback stabilizable if there exists a matrix $K(i,j)$ such that the system of equation 13 is asymptotically stable [see def 4.1].

Therefore definition 2 insures that the states are getting smaller as n increases. Similar to chapter 4 the matrix

$$\bar{\Phi}(n;2) = \{[A(n)+B(n)K(n)][A(n-1)+B(n-1)K(n-1)]\}^T$$

$$\cdot\{[A(n)+B(n)K(n)][A(n-1)+B(n-1)K(n-1)]\} \qquad (15)$$

can be analyzed. It should be noted that the matrix $\bar{\Phi}(n;2)$ has the same structure as that of $\Phi(n;2)$. Therefore all theory available in chapter 4 can be extended to this case. The m-step state feedback stabilization can be discussed similarly. Due to the identical structure of the matrices $\bar{\Phi}(n;i)$ and $\Phi(n;i)$, the equivalent extended results can be stated. In the interests of brevity we will not state such results here.

The remaining question is how one can find the matrix $K(i,j) = [K_1 \quad K_2](i,j)$ such that $\sigma_{max}\bar{Q}(i,j) < 1$. Since the matrix $\bar{Q}$ is a non linear function of $K(i,j)$, choosing $K(i,j)$ to yield a specific set of eigenvalues is analytically very difficult. However, if an initial $K(i,j)$ is chosen, then an associated maximum eigenvalue of the matrix $\bar{Q}$ can be found. Several algorithms have been developed for computing the eigenvalues of a fixed matrix [70] and [76]. If the maximum eigenvalue of $\bar{Q}$ is less than one, the system is asymptotically stable. If the maximum eigenvalue exceeds one, it is feasible to iterate using, perhaps a gradient procedure. It is apparent that the above procedure is not efficient. Another approach based in the Lyapunov criteria will be discussed in the following.

Lyapunov Methods: Considering the time invariant case, a state transition equation, with system matrix $\bar{Q}$ is shown to be asymptotically stable if and only if there exists $P > 0$ and $R < 0$ satisfying the equation

$$\bar{Q}^T P \bar{Q} - P = -R.$$

By using the Kronecker product the above matrix equation can be put in vector form. In short one obtains the following equation

$$\{[(A+BK)(A+BK)^T]\otimes[(A+BK)(A+BK)^T]-I\otimes I\}\text{vec}(p) = -\text{vec}(q)$$

where $\text{vec}(p) = \text{col}[p_{11}, p_{12}, \cdots p_{1n}, p_{21}, \cdots p_{2n}, \cdots\cdots p_{nn}]$ and vec(q) is similarly defined. Now solve for vec(p). The above criteria is applied by assuming a qualifying matrix R and determining the companion P. The search for a P, R pair satisfying the above criteria involves, in general, repetitious trial and error.

However, we can obtain an alternative result by considering equation 4.36' and applying theorem 4.19. This simplifies the solution since we consider the matrix (A+BK) instead of $\bar{Q} = (A+BK)^T(A+BK)$. Note that in this case the matrix P(n) is assumed to be block diagonal.

Applying equation 4.36' to (A+BK) yields

$$(A+BK)^T P(A+BK) - P = -Q.$$

By some manipulations the following Riccati equation can be obtained

$$(A^T PB)K + K^T(B^T PA) + K^T(B^T PB)K + (Q-P+A^T PA) = 0 \qquad (16)$$

where matrix K has dimension p x (n+m), P is the number of inputs and (n+m) is the size of the state matrix A.

Now the following result is immediate.

<u>Result</u>: The system 1 is stabilizable by using state feedback K if there exists a positive definite matrix $P = P_1 \oplus P_2$, Q such that algebraic Riccati equation 16 has

a real solution K.

Conditions for existence and uniqueness of equation 16 are discussed in [77-79]. This has been done by constructing the Hamiltonian matrix.

In order to be able to use theorems 4.8 and 4.9 with consideration of equation 4.19 it is necessary to know if there exists a matrix $K(i,j)$ such that $[A+BK](i,j)$ is symmetric. Assuming there exists such a $K(i,j)$ it is possible to compute the maximum eigenvalues of the matrix $[A+BK](i,j)$. If its maximum eigenvalue is not less than one, another qualified $K(i,j)$ is chosen. The maximum eigenvalue of the above matrix is determined. The procedure is continued until a desired $K(i,j)$ is found. Necessary and sufficient conditions for the existence of $K(i,j)$ such that $[A+BK](i,j)$ is symmetric will be given in the following.

5.2.2 Necessary and Sufficient Conditions for the Existence of K(i,j)

It is necessary to find $K(i,j)$ such that $[A+BK]^T(i,j) = [A+BK](i,j)$. Here A is n x n, B is n x p and K is p x n. For simplicity in the operation we drop, temporarily, the dependence on the indices (i,j). We must have

$$a_{rm} + (BK)_{rm} = a_{mr} + (BK)_{mr}, \qquad r \neq m,\ r > 1,\ m < n.$$

This set of $n(n-1)/2$ equations can be written in matrix form. For this we use the following notation

k_m = m-th column of the matrix K, $m = 1,2,\ldots, n$;

$\overline{K} = \text{col}\ (k_1, k_2, \ldots\ k_p)$;

$\bar{a} = \text{col}\,(a_{21}-a_{12},\ldots a_{n1}-a_{1n},\ a_{32}-a_{23},\ldots\ldots,\ a_{n,n-1}-a_{n-1,n})$;

b_r^t = r-th row of the B matrix, $r = 1,2,\ldots., n$.

The system of equations becomes

$$D\bar{K} = \bar{a}. \tag{17}$$

In partitioned form, the matrix D has the following value

$$D = \begin{bmatrix} -b_2^t & b_1^t & 0 & 0\ldots\ldots\ldots 0 & \\ -b_3^t & 0 & b_1^t & 0\ldots\ldots\ldots 0 & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & \ddots \\ -b_n^t & 0\ldots\ldots\ldots\ldots\ldots 0 & & & b_1^t \\ 0 & -b_3^t & b_2^t & 0\ldots. 0 & \\ & \vdots & & \ddots & \\ & \vdots & & & \ddots \\ 0 & -b_n^t & 0\ldots\ldots\ldots 0 & & b_2^t \\ & & & & \\ 0 & 0\ldots. 0 & & -b_{n-1}^t & b_n^t \end{bmatrix}.$$

Since the matrix D and the vectors $\bar{K}$, $\bar{a}$ are functions of (i,j) we state the following lemma accordingly.

<u>Lemma(2)</u>: The matrix K(i,j) can be chosen such that [A+BK](i,j) is symmetric iff

$$\text{rank}\ [D](i,j) = \text{rank}\ [D : \bar{a}](i,j) \tag{18}$$

<u>Proof</u>: The matrix [A+BK](i,j) is symmetric iff $[D\bar{K}](i,j) = \bar{a}(i,j)$. Clearly range $\bar{a}(i,j)$ is a subset of range D(i,j) and hence the condition of equation 18 holds.∎

<u>Remark 1</u>. In order to find K such that (A+BK) is symmetric it is needed to solve the set of n(n-1)/2 linear equations.

If the matrix D is nonsingular (which is not possible for single-input system) then K always exists.

Note that D has dimension $n(n-1)/2 \times np$ for a multi-input system, where p is equal to number of inputs. Thus D could be nonsingular iff $n(n-1)/2 = np$ (i.e $p = (n-1)/2$). Also note that if $p > (n-1)/2$, a solution may exist but may not necessarily be unique.

Remark 2. If D does not have an inverse we may want to find a DRAZIN or PENROSE inverse to see how small the non-symmetric part is and observe what its effect on stability is. This is a potential topic of future research.

Assuming no matrix K can be found in order to make [A+BK] symmetric, the next question is what can be said if the matrix [A+BK] is symmetrizable. The following section considers this case.

5.2.3 Similarity Transformation

In this section we restrict attention to the stationary case. We discuss two alternative methods. The first method considered the block diagonal matrix T. This preserves the eigenvalues of the 2-D system. The second method is based on the use of a unitary matrix. Under this transformation the norm and eigenvalues are invariant. By the use of corollary 4.3 and theorem 4.10 the following theorems can be stated respectively.

Theorem (1): The system 1 is one-step state feedback stabilizable if the following conditions are satisfied:

i) There exists a matrix $K = [K_1 \quad K_2]$ such that

$|\sigma_{max}F| < 1$, where $F = A+BK$

ii) There exists a nonsingular block diagonal matrix T such that $E = TFT^{-1}$ is symmetric.

Proof: Assume there exists a matrix K such that matrix F is asymptotically stable. Therefore E is also asymptotically stable since by the similarity transformation the characteristic polynomial of F is invariant. On the other hand E is symmetric. Thus by corollary 4.3 system 3 is asymptotically stable. Using definition 1 implies that system 1 is one-step state feedback stabilizable.■

Theorem (2): The system 1 is one-step state feedback stabilizable if the following conditions are satisfied:

i) There exists a matrix $K = [K_1 \quad K_2]$ such that
$|\sigma_{max} F| < 1$, where $F = A+BK$

ii) There exists a unitary (i.e. $T^{-1} = T^T$) matrix T such that $E = TFT^{-1}$ is symmetric.

Proof: Assume there exists a unitary matrix T such that the matrix E is symmetric. Then $||F|| = ||E|| = |\sigma_{max}(E)| = |\sigma_{max}(F)|$. Thus by condition i) $||F|| < 1$. Therefore by theorem 4.10 the system 3 is asymptotically stable. Using definition 1 implies that system 1 is one-step state feedback stabilizable.■

The theorem suggests the following procedure to verify stability:

(1) Find matrix K such that $F = A+BK$ is asymptotically stable.

(2) Find matrix T such that $E = TFT^{-1}$ is symmetric.

It should be noted that if either condition 1 or 2 of the above procedure is not satisfied the theorems 1 and 2 fail to be applicable. The problem of determining K has been exhaustively studied in the context of 1-D state feedback. See for instance the references [73,80].

Remark 3. In order to find T we need to solve a set of linear equations. The number of unknowns is equal to $(n+m)^2$ where $(n+m)$ is the dimension of the system. The total number of equations is equal to $(n+m)(n+m-1)/2$. Obviously, the number of equations is less than the number of unknowns; that is $\{(n+m)^2+(n+m)\}/2$ degrees of freedom exist.

Remark 4. If in the previous procedure a matrix T is not found, we then need to choose another K such that step 1 is satisfied and then apply step 2 again. This is very time consuming. However, in [81] it has been shown how to characterize the class of K such that matrix F is stable.

5.3 OUTPUT FEEDBACK STABILIZATION

In some cases the state vector may not be measurable. In these cases, control through output feedback, rather than by state feedback, would have to be considered.

In this section we attempt to stabilize the system by using output feedback rather than by state feedback. In this case the following equation for the closed-loop system is obtained

$$\phi(n+1) = [A(n)+B(n)R(n)C(n)]\phi(n)$$
$$= T_2(n) \qquad (19)$$

Def (3): The system of equation 1 is said to be one-step output feedback stabilizable if there exists a matrix $R(n)$ such that the system of equation 19 is asymptotically stable [see def 4.1].

Now the following corollaries are immediate.

Corollary (3): The system of equation 19 is aymptotically stable if

$$\prod_{i=k}^{p} ||A(i) + B(i)R(i)C(i)||$$

goes to zero as p goes to ∞.

Proof: The proof of this corollary follows directly from theorem 4.1. This is due to the fact that $A(n)$ and $[A(n)+B(n)R(n)C(n)]$ have an identical structure.■

Corollary (4): The system of equation 19 is aymptotically stable if $||A(i) + B(i)R(i)C(i)|| < 1-\varepsilon$ for $i= 1,2,3 \ldots$ where $\varepsilon > 0$.

Proof: Similar to the proof of above corollary but using the theorem 4.2.■

Comparison: In the case of 1-D, for output feedback stabilization it is required to determine a matrix R such that the $\sigma_{max}(A+BRC)$ is less than one. There are many ways to solve this problem. For instance, by the use of Kronecker product this is equivalent to solving a system of equations $Pr = q$ where $P = C^T \otimes B$, $r = vec(R)$, $q = vec(Q)$ [74]. It is well known that above equation has a solution if and only if $rank[P] = rank[P:q]$. For 2-D case the goal is to find R such that the maximum eigenvalue of $\bar{\bar{Q}} = [A+BRC]^T[A+BRC]$ is less than one. Note that the matrix $\bar{\bar{Q}}$ is a nonlinear function of R and it has a Ricatti form. Thus it is not possible to state necessary and sufficient conditions for existence of a solution for R. As we have seen, stating the sufficient condition for stabilizing the system is not difficult. However checking that condition is tedious.

5.4 BOUNDARY CONTROL

It is of interest to determine the effect of boundary feedback control on a given system. Consider the nonstationary Wave model summarized in equation 1. Let the boundary control , f(n), be generated by the state feedback law

$$f(n) = L(n)\phi(n) \tag{20}$$

where L(n) is defined in equation 6. Then we obtain the state transition equation as follows

$$\begin{aligned}\phi(n+1) &= [A(n) + E(n)L(n)]\phi(n) \\ &= T_3(n)\phi(n). \end{aligned} \tag{21}$$

Note that only the first block and the last block of $T_3(n)$ are affected by boundaries. The rest of the blocks remain the same as blocks of A(n). In the following section, we explore the partial effect of boundary control on the stability of the Wave model.

5.4.1 Stability Analysis

Consider the system defined by equation 21. Using theorem 4.8 the obtained system 21 is asymptotically stable if $||A(n)+E(n)L(n)|| < 1-\varepsilon$ for $\varepsilon > 0$ and all n. We are now interested in knowing if there exists a matrix L(n) in order to make system 21 asymptotically stable. From the previous discussion the above condition can be satisfied if $|\sigma_{max}[T^T{}_3(n)T_3(n)]| < 1-\varepsilon$. Thus we need to determine the eigenvalues of the matrix $T^T{}_3(n)T_3(n)$. From result 4.1 we have

$$\sigma[T^T{}_3(n)T_3(n)] = \sigma\bar{M}(n,0) \cup \sigma\bar{Q}(n-j,j) \cup \sigma\bar{N}(0,n) \qquad (22)$$

for any n and $j = 1,2,\ldots, n-1$. Even though the matrices $\bar{M}$ and $\bar{N}$ are specified in terms of $L(n)$, the matrix $\bar{Q}$ is independent of $L(n)$. Thus its eigenvalues cannot be affected by boundary control. In addition, it has been proven in theorem 4.10 that for the stationary case the maximum eigenvalue of the matrix Q determines the stability of the system. Therefore we have shown:

Theorem (3): boundary control does not affect one step stability of the system.

Other conditions for stability could also be applied. For instance, when using the 2-step test, the following analysis should be done. Equation 21 can be modified to the form

$$\phi(n) = [A(n-1) + E(n-1)L(n-1)]\phi(n-1). \qquad (23)$$

Combining equations 21 and 23 yields

$$\phi(n+1) = [A(n) + E(n)L(n)][A(n-1) + E(n-1)L(n-1)]\phi(n-1) \quad (24)$$

The eigenvalues of the matrix $[A(n)+E(n)L(n)][A(n-1)+E(n-1)L(n-1)]^T[A(n)+E(n)L(n)][A(n-1)+E(n-1)L(n-1)]$ should be analyzed. This can be easily extended to the m-step case.

Similar discussion can be made where $f(n) = S(n)\mu(n)$. In the interests of brevity we will not state such results here.

In the following we present several examples to demonstrate the use of state feedback.

Example (1): Given an unstable system of equation 4.2 with the following matrices;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

it follows easily that

$$A+BK = \begin{bmatrix} 1+K_1 & 1+K_2 \\ -K_1 & -2-K_2 \end{bmatrix}$$

One way to stabilize the system is to choose K_1 and K_2 such that the matrix A+BK is block triangular. This implies $K_1 = 0$ or $1+K_2 = 0$. For the first case (i.e. $K_1=0$) the matrix A+BK has always a unity eigenvalue. Thus it can not be stabilizable. For the second case the matrix A+BK has also an eigenvalue of magnitude one. Thus it is not possible to stabilize the system.

In order to apply the theory presented in section 5.2 we should determine if there exists a K such that the matrix A+BK is symmetric. It can easily be verified that rank[D] = rank[D:$\bar{a}$]. Thus by lemma 2 a solution exists. The set of solutions should satisfy the equation $1+K_2 = -K_1$. This implies that the matrix A+BK is of the form

$$A+BK = \begin{bmatrix} 1+K_1 & -K_1 \\ -K_1 & K_1-1 \end{bmatrix}$$

It can easily be verified that

$$|\sigma_{max}(A+BK)| \geq \max_{K_1} \{|1+K_1|, |K_1-1|\} \geq 1$$

Thus the system of equation 4.2 is not stabilizable.

Example (2): Consider the system of equation 4.2 with the following matrices;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

it follows easily that

$$A+BK = \begin{bmatrix} 1+K_1 & 1+K_2 \\ -K_1 & -1-K_2 \end{bmatrix}$$

Method 1

Similar to example 1, if $K_1=0$ (i.e. $A_3+B_2K_1=0$) then the system is not stabilizable. However if $K_2=-1$ (i.e. $A_2+B_1K_2=0$) then the system is stabilizable. For this case $K_2=-1$ and $-2<K_1<0$.

Method 2

This method takes advantage of theory available in section 2 of this chapter. It can easily be verified that $D = [-1 \;\; -1]$, $B = [-1]$. Thus $\text{rank}[D] = \text{rank}[D:\bar{a}]$. Therefore there exists a matrix K to make A+BK symmetric. The set of solution K should satisfy the equation $1+K_2 = -K_1$. For this case the characteristic equation of A+BK is

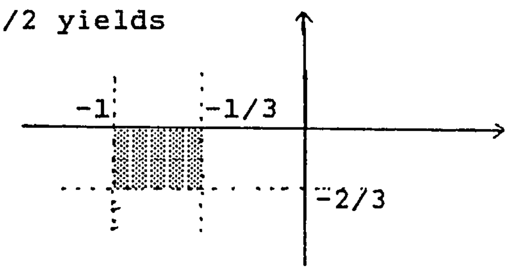
$$s^2 - (K_2-K_1)s + K_1 = s^2 - (1+2K_1)s + K_1 = 0 \qquad (25)$$

From [73] the eigenvalues of equation 25 are inside the unit circle if and only if the following inequalities hold:

$$K_1 < 1$$
$$K_1 > -1 + (-1-2K_1)$$
$$K_1 > -1 - (-1-2K_1)$$

These inequalities imply that $-2/3 < K_1 < 0$. This yields that $-1 < K_2 < -1/3$. Thus by choosing any K in the shaded area below the system can be stabilized. In particular choosing $K_1 = -1/2$ and $K_2 = -1/2$ yields

$$A+BK = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$



which has been shown to be stable (see example 4.1).

<u>Method 3</u>

In this method the solution of K is sought by solving the Riccati equation 16. It can easily be verified that (see also [63] for more details) with the choice of P = I and Q = (1/2)I the Riccati equation has a real solution K = [-1/2 -1/2].

CHAPTER 6

STATE OBSERVER OF THE WAVE MODEL

ABSTRACT

Results concerning the state estimation of discrete-time linear time invariant 2-D systems described by a Givone-Roesser (G-R) model [2] are presented. The proposed method avoids the assumption that the observer (estimator) characteristic polynomial is separable. The observer matrix is chosen to be symmetric such that the characteristic polynomial of the observer is a Shank's polynomial [11]. The problem of estimating the state of the 1-D Wave model established by Porter-Aravena [6] as equivalent to the 2-D (G-R) model is also considered. Two different methods are stated. Some examples are provided to illustrate the technique.

6.1 INTRODUCTION

A substantial portion of the optimal control theory relies on the use of feedback, acting on the state variables to generate the control. In practice, however, not all of the state variables are usually accessible. Therefore, when feedback from all the state variables is required in a given design, and not all the state variables are accessible, it is necessary to 'synthesize' the states from information contained in the output as well as the input variables. The subsystem that performs this function from the measurements of the input and the output is called a state observer, or simply an observer.

Figure. 1 shows the block diagram of a digital control system which has a state observer. In general, the state observer shown in Fig.1 is to be designed so that the observed state will be as close as possible to the actual state. There are many ways of designing a digital state observer, and generally there is more than one way of judging the closeness of state observer to actual state. Intuitively, the state observer should have the same state equations as the original system. We must first establish the condition under which an effective observer exists.

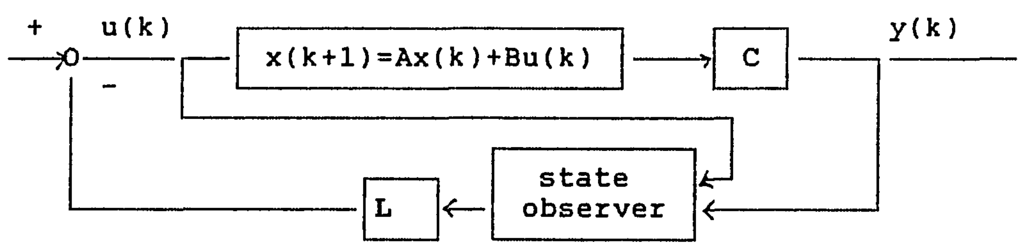


Figure 1

Since the original work of Luenberger on the design of the observer for 1-D systems, many authors have considered various design methods for state and functional observers. Unfortunately there is no unifying framework as such for 2-D systems.

First we consider the problem of estimating the state of nonstationary 2-D systems for the discrete case described by G-R model (see equation 4.1).

The method used by [82] leads to a design of an observer with a diagonal observer matrix for the stationary case, which forces the state matrix A to be diagonal. This implies a strict necessary condition on the original system. In [83] it is assumed that the observer matrix has a separable characteristic polynomial. Clearly this is more general than [82]. With the separability assumption the single 2-D system is transformed into two simultaneous 1-D systems and therefore avoids the difficulties inherent with the stability of 2-D systems [84,85]. We improve on these previous results by requiring the observer matrix to be symmetric. In this way separability is no longer a necessary condition. The Shank's polynomial [11] is generated using our corollary 4.3.

The second part of our work is based on considering the nonstationary 1-D Wave model presented in equation 4.2. The fact that the state matrix A(n) is not a square matrix implies that the estimator matrix is not a square matrix either. Therefore it is not even possible to discuss its eigenvalues for the stationary case. However we are able to discuss the norm of the state estimator in order to discuss the rate of convergence of state to its estimator.

6.2 OBSERVER THE FOR G-R MODEL

In this section we extend 1-D observer theory to the 2-

D case. For this we consider the nonstationary G-R model of equation 4.1. Two methods are discussed.

6.2.1 Method I

The equation of the state estimator is defined in the following

$$\begin{bmatrix} z^h(i+1,j) \\ z^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1(i,j) & A_2(i,j) \\ A_3(i,j) & A_4(i,j) \end{bmatrix} \begin{bmatrix} z^h(i,j) \\ z^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1(i,j) \\ B_2(i,j) \end{bmatrix} u(i,j)$$

$$+[L_1(i,j) \quad L_2(i,j)]\{y(i,j)-[C_1(i,j) \quad C_2(i,j)] \begin{bmatrix} z^h(i,j) \\ z^v(i,j) \end{bmatrix}\}$$

$$= [A-LC](i,j)z+B(i,j)u+L(i,j)y \tag{1}$$

Define

$$e(i, j) = x(i, j) - z(i, j)$$

then

$$\begin{bmatrix} e^h(i+1, j) \\ e^v(i, j+1) \end{bmatrix} = [A-LC](i,j) \begin{bmatrix} e^h(i, j) \\ e^v(i, j) \end{bmatrix} \tag{2}$$

if the matrix $L(i,j)$ can be chosen such that the system equation 2 is asymptotically stable then the error converges to zero. By use of theorem 4.8 the following theorems can be stated.

<u>Theorem (1):</u> The system equation 4.1 can be estimated by state estimation 1 if there exists a matrix $L(i,j)$ such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_Q(n)$$

goes to zero, where $Q = [A-LC]^T[A-LC]$.

By the use of similarity transformation the following theorems are immediate.

<u>Theorem (2):</u> The system equation 4.1 can be estimated by state estimation 1 if the following holds

1) there exists a matrix $L(i,j)$ such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_{[A-LC]}(n)$$

goes to zero.

2) there exists a unitary matrix $T(i,j)$ such that the matrix $[T(A-LC)T^{-1}](i,j)$ is symmetric

<u>Remark</u>. It should be noted that for both of the above theorem the stability could be also insured by the use of theorem 4.9.

For the stationary case the stability can be discussed in terms of the eigenvalues. Using corollary 4.3 gives the following result.

<u>Theorem (3):</u> The system equation 4.1 can be estimated by state estimation 1 if the following holds

1) there exists a matrix L such that $|\sigma_{max}[A-LC]| < 1$.
2) there exists a nonsingular matrix T such that the matrix $T(A-LC)T^{-1}$ is symmetric where

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

Proof: The matrices $T(A-LC)T^{-1}$ and $(A-LC)$ have the same eigenvalues by use of similarity transformation. By corollary 4.3 the system with state matrix $T(A-LC)T^{-1}$ is asymptotically stable. Thus the system equation 2 is also asymptotically stable.

Theorem (4): The system equation 4.1 can be estimated by state estimation 1 if the following holds

1) there exists a matrix L such that $|\sigma_{max}(A-LC)| < 1$.
2) there exists a unitary matrix T such that the matrix $[T(A-LC)T^{-1}]$ is symmetric.

6.2.2 Method II

Define the full observer

$$\begin{bmatrix} z^h(i+1,j) \\ z^v(i,j+1) \end{bmatrix} = \begin{bmatrix} F_1(i,j) & F_2(i,j) \\ F_3(i,j) & F_4(i,j) \end{bmatrix} \begin{bmatrix} z^h(i,j) \\ z^v(i,j) \end{bmatrix} + \begin{bmatrix} G_1(i,j) \\ G_2(i,j) \end{bmatrix} u(i,j)$$

$$+ \begin{bmatrix} D_1(i,j) \\ D_2(i,j) \end{bmatrix} y(i,j) \qquad (3)$$

where $z^h \varepsilon R^n$, $z^v \varepsilon R^m$.

The state $z(i,j)$ in 3 is an estimate of $x(i,j)$ in 4.1 if $e(i,j) = z(i,j) - x(i,j)$ goes to zero when i, j goes to ∞ independent of input $u(i,j)$ and for arbitrary boundary conditions. Now we can state the following theorem:

<u>Theorem (5):</u> The system defined by 3 is a full order observer for the system given by 4.1 if the parameters of system 3 satisfy the following:

(i) There exists a matrix L(i,j) such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_Q(n)$$

goes to zero.

(ii) G(i,j) = B(i,j)

(iii) D(i,j) = L(i,j)

<u>Theorem (6):</u> The system defined by 3 is a full order observer for the system given by 4.1 if a matrix T(i,j) and the parameters of system 3 satisfy the following:

(i) There exists a matrix L(i,j) such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_{[A-LC]}(n)$$

goes to zero.

(ii) There exists a unitary matrix T(i,j) such that the matrix $[T(A-LC)T^{-1}](i,j)$ is symmetric

(iii) G(i,j) = B(i,j)

(iv) D(i,j) = L(i,j)

For the stationary case we obtain the following theorem.

<u>Theorem (7):</u> The system defined by 3 is a full order observer for the system given by 4.1 if a matrix T and

the parameters of system 3 satisfy the following:

(i) There exists a matrix $L = [L_1 \quad L_2]^T$ such that $F = A - LC$ has a maximum eigenvalue less than unity.

(ii) There exists nonsingular matrix T such that $E = TF^{-1}T$ is symmetric where

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

(iii) $G = B$

(iv) $D = L$

<u>Proof</u>: Consider the error vector $e(i,j) = z(i,j) - x(i,j)$ and also $\hat{e} = Te$. If (iii) and (iv) are satisfied we find that

$$\begin{bmatrix} e^h(i+1, j) \\ e^v(i, j+1) \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} e^h(i, j) \\ e^v(i, j) \end{bmatrix}.$$

and

$$\begin{bmatrix} \hat{e}^h(i+1,j) \\ \hat{e}^v(i,j+1) \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{e}^h(i, j) \\ \hat{e}^v(i, j) \end{bmatrix}$$

By the similarity transformation the characteristic polynomial of F is invariant. Thus by corollary 4.3 $e(i,j)$ is bounded as i,j goes to ∞. The result leads to the following:

<u>Algorithm:</u>

(1) Find L such that $F = A - LC$ is asymptotically

stable (i.e. maximum eigenvalue of matrix F is less than one).

(2) Find matrix T such that $E = TF^{-1}T$ is symmetric.

It should be noted that if the pair (A,C) is observable we always can find L. There exists many algorithms for finding L.

<u>Remark.</u> For finding the matrix T we need to solve a set of equations. The number of unknowns is equal to $(n^2 + m^2)$ where n^2, m^2 are number of elements of the matrices T_1 and T_2 respectively. The total number of equations is equal to $(n+m)(n+m-1)/2$. Clearly, the number of equations are less than the number of unknowns; that is we have $k = [(n-m)^2 + (n+m)]/2$ degrees of freedom.

<u>Remark.</u> If T is not found in step 2, we then need to choose another L such that step 1 is satisfied and then apply step 2 again. Unfortunately there is no apriori check on the existence of L and T.

<u>Example</u> Given the following matrices for G-R model

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -5 \end{bmatrix} , \qquad C = [1 \ 0].$$

Note that this system is unstable because $|\sigma_{max}(A_4)| > 1$. Let's compute $|zI - F| = |zI - (A-LC)| = z^2 + (L_2+5)z + (L_1+2)$. Choose $L = [-39/20 \quad 44/10]^T$ then $z = -1/5$, $z = -1/10$ that is maximum eigenvalue of matrix F is less than one. However we can not find any matrix T such that condition 2 is satisfied. By choosing $L = [-41/20 \quad -44/10]^T$ the matrix

T = diag (20,11) will be found. Therefore we will have

$$E = TF^{-1}T = \begin{bmatrix} 0 & 1/20 \\ 1/20 & -4/10 \end{bmatrix}.$$

with eigenvalues 1/2, -1/10. For this choice of L both steps are satisfied.

Note that the 2-D characteristic polynomial $p(z^{-1},w^{-1}) = 1 + 0.4\ w^{-1} - 0.05\ z^{-1}w^{-1}$ is stable [18].

6.3 <u>OBSERVER FOR THE WAVE MODEL</u>

In this case the nonstationary Wave model is considered. The 1-D theory is applied to the Wave model, in order to derive an observer for the 2-D system of equation 4.2.

6.3.1 <u>Method I</u>

Suppose the equation of the state estimator is given by

$$\theta(n+1) = A(n)\theta(n) + B(n)v(n) + E(n)f(n) + L(n)[\mu(n) - D(n)v(n) - H(n)f(n) - C(n)\theta(n)] \qquad (4)$$

Define $\quad e(n) = \phi(n) - \theta(n).$

Then

$$\begin{aligned} e(n+1) &= \phi(n+1) - \theta(n+1) \\ &= A(n)\phi(n) - A(n)\theta(n) - L(n)C(n)\phi(n) + L(n)C(n)\theta(n) \\ &= [A(n) - L(n)C(n)]\, e(n) \\ &= W(n)e(n) \qquad (5) \end{aligned}$$

equation 5 is a homogeneous state equation as is to be expected from the theory of observer. However, in the present case [A(n)-L(n)C(n)] is not stationary and in fact increases in dimension systematically.

Now we need to find the structure of L(n). It is clear that banded structure of A(n) is required for quarter plane causality. C(n) is also given in a special form. If we want the observer of equation 4 to have quarter plane causality the form of L(n) has to be chosen such that L(n)C(n) has the same format of A(n). After some manipulations we can see that L(n) must have the following banded format:

$$L(n) = \begin{bmatrix} L_2(n,0) & 0 & & 0 \\ L_1(n,0) & 0 & & 0 \\ 0 & L_2(n-1,1) & & \\ \cdot & L_1(n-1,1) & & \\ \cdot & \cdot & & \\ & & \cdot & \\ 0 & & & L_2(0,n) \\ 0 & & & L_1(0,n) \end{bmatrix} \tag{6}$$

and therefore the matrix W(n) will be of the form

$$W(n) = \begin{bmatrix} \hat{A}_4(n,0) & 0 & 0 & 0 & 0 \\ \hat{A}_2(n,0) & 0 & 0 & 0 & 0 \\ 0 & \hat{A}_3(n-1,1) & \hat{A}_4(n-1,1) & 0 & 0 \\ 0 & \hat{A}_1(n-1,1) & \hat{A}_2(n-1,1) & 0 & \\ 0 & & & & \\ \vdots & \cdot & & & \\ \vdots & & \cdot & & \\ \vdots & & \cdot & & \\ \vdots & & & \cdot & \\ \vdots & & & & \\ 0 & 0 & 0 & & \hat{A}_3(0,n) \\ 0 & 0 & 0 & & \hat{A}_1(0,n) \end{bmatrix} \tag{7}$$

where

$\hat{A}_1(k,l) = [A_1 - L_2C_1](k,l)$

$\hat{A}_2(k,l) = [A_2 - L_2C_2](k,l)$

$$\hat{A}_3(k,l) = [A_3 - L_1C_1](k,l)$$
$$\hat{A}_4(k,l) = [A_4 - L_1C_2](k,l).$$

From 1-D observer theory it is possible to get a valid estimator for the state if there exist a matrix L(n) such that the system equation 5 is asymptotically stable. Note that the size of the state matrices increases as n increases and moreover [A(n)-L(n)C(n)] is not a square matrix. Thus the spectral considerations are not pertinent. However, we can consider matrix norm criteria.

Def(1): Equation 4 is a valid estimate of equation 4.2 if there exists a matrix L(n), as in equation 6, such that system 5 is asymptotically stable [see def 4.1].

Since [A(n) - L(n)C(n)] has the same structure as [A(n) + B(n)K(n)], all corollaries that were proved in chapter 5 regarding state feedback, are also valid in the present setting. For adaptation to this chapter only the substituting [A(n) - L(n)C(n)] for [A(n) + B(n)K(n)] is necessary. For reader's convenience we state some of these extensions.

Corollary (1): the state $\theta(n)$ is an estimate of $\phi(n)$ in the sense that $[\phi(n) - \theta(n)]$ goes to zero as n goes to ∞ if there exists a matrix L(n) such that

$$\prod_{n=1}^{p} ||A(n)-L(n)C(n)||$$

goes to zero as p goes to ∞.

Proof: The proof of this corollary follows directly from theorem 4.1 and definition 1. This is due to the fact that $A(n)$ and $[A(n) - L(n)C(n)]$ have an identical structure.■

<u>Corollary (2)</u>: the state $\theta(n)$ is an estimate of $\phi(n)$ in the sense that $[\phi(n) - \theta(n)]$ goes to zero as n goes to ∞ if there exists a matrix $L(n)$ such that

$$||A(i) - L(i)C(i)|| < 1-\varepsilon \quad \text{for } i= 1,2,3 \ldots \quad \text{where } \varepsilon > 0.$$

<u>Proof</u>: Similar to the proof of above corollary but using the theorem 4.2.■

Note that it is not possible to state any necessary and sufficient condition for existence of observer. This is due to the fact that the norm of a matrix gives only sufficient condition for stability.

It can easily be seen that the matrix $\Phi(n) = W^T(n)W(n)$ has the form

$$\hat{\Phi}(n) = \begin{bmatrix} \hat{M}(n,0) & 0 & 0 & 0 \ldots\ldots\ldots 0 & \\ 0 & \hat{Q}(n-1,1) & 0 & 0 \ldots\ldots\ldots 0 & \\ 0 & & \hat{Q}(n-2,2) & & \\ 0 & & \cdot & & \\ \cdot & & & \cdot & \\ 0 & & & & \\ 0 & & & \hat{Q}(1,n-1) & 0 \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots 0 & & & & \hat{N}(0,n) \end{bmatrix} . \tag{8}$$

The matrix $\hat{Q}$ has the form

$$\hat{Q}(k,l) = \begin{bmatrix} \hat{M}(k,l) & \hat{P}(k,l) \\ \hat{P}^T(k,l) & \hat{N}(k,l) \end{bmatrix}, \tag{9}$$

where

$$\hat{M}(k,l) = \hat{A}_4{}^T(k,l)\hat{A}_4(k,l)+\hat{A}_2{}^T(k,l)\hat{A}_2(k,l)$$

$$\hat{N}(k,l) = \hat{A}_3^T(k,l)\hat{A}_3(k,l)+\hat{A}_1^T(k,l)\hat{A}_1(k,l)$$
$$\hat{P}(k,l) = \hat{A}_3^T(k,l)\hat{A}_4(k,l)+\hat{A}_1^T(k,l)\hat{A}_2(k,l).$$

It should be noted that the matrix $\hat{\Phi}(n)$ is block diagonal. Thus similar to the state feedback case we obtain

$$||W(n)|| = \sigma_{max}{}^{1/2}\hat{\Phi}(n) = \max_j \{\sigma_{max}{}^{1/2}\hat{Q}(n-j,j) : j=0,1,..,n\}. \tag{10}$$

The feedback case equation 4.17-4.19 are modified to

$$\text{(i)} \quad \bar{\sigma}_Q(n) = \max_j \{\sigma_{max}{}^{1/2}\hat{Q}(n-j,j) : j=0,1,..,n\}.$$

$$\text{(ii)} \quad \bar{\sigma}_Q(n) = \max_j \{||[A-LC](n-j,j)|| : j=0,1,..,n\}.$$

$$\text{(iii)} \quad \bar{\sigma}_Q(n) = \max_j \{|\sigma_{max}[A-LC](n-j,j)| : j=0,1,..,n\}. \tag{11}$$

for the stationary case

$$\bar{\sigma}_Q(n) = \sigma_{max}{}^{1/2}(\hat{Q}) = ||A-LC|| \tag{12}$$

and if (A-LC) is symmetric

$$\bar{\sigma}_Q(n) = |\sigma_{max}(A-LC)|. \tag{13}$$

Note that $\bar{\sigma}_Q(n)$ is in terms of the state matrix A of the G-R model. Therefore considering the G-R model similar corollaries can be stated. This also agrees with the result that we have obtained in section 2.

With appropriate choice of $\bar{\sigma}_Q(n)$, corollaries 1 and 2 are respectively modified as follows

<u>Corollary (3)</u>: the state $\theta(n)$ is an estimate of $\phi(n)$ in the sense that $[\phi(n) - \theta(n)]$ goes to zero as n goes to ∞ if

there exists a matrix L(n) such that

$$\prod_{n=1}^{p} \bar{\sigma}_Q(n)$$

goes to zero as p goes to ∞.

<u>Corollary (4)</u>: the state $\theta(n)$ is an estimate of $\phi(n)$ in the sense that $[\phi(n) - \theta(n)]$ goes to zero as n goes to ∞ if there exists a matrix L(n) such that

$$\bar{\sigma}_Q(i) < 1-\varepsilon \quad \text{for } i= 1,2,3 \ldots \quad \text{where } \varepsilon > 0.$$

In order to apply corollaries 3 and 4 with consideration of equation 13, the existence of L(i,j) such that $[A-LC]^T(i,j) = [A-LC](i,j)$ is required. This procedure has already been discussed in relation to state feedback (see 5.2.2). No further analysis is necessary here.

6.3.2 <u>Method II</u>

Suppose that we have a Wave model described by equation 4.2. If C(n) is nonsingular, the state vector $\phi(n)$can easily be determined from $\phi(n) = C(n)^{-1} [\mu(n)-D(n)v(n)-H(n)f(n)]$. If that is not possible we let the output vector $\mu(n)$ serve as the input to a linear observer described by

$$\theta(n+1) = G(n)\theta(n) + N(n)v(n) + M(n)f(n) + L(n)\mu(n) \qquad (15)$$

where G(n) has the same structure of A(n) and

$$N(n) = \begin{bmatrix} N_1(n,0) & 0 & & 0 \\ N_2(n,0) & 0 & & 0 \\ 0 & N_1(n-1,1) & & \\ \cdot & N_2(n-1,1) & & \\ \cdot & & \cdot & \\ & & & \cdot \\ 0 & & & N_1(0,n) \\ 0 & & & N_2(0,n) \end{bmatrix}$$

$$M(n) = \begin{bmatrix} M_3(n,0) & 0 \\ M_1(n,0) & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdot & \cdot \\ & M_4(0,n) \\ 0 & M_2(0,n) \end{bmatrix}.$$

We assume the matrices have appropriate dimensions. We shall attempt to adjust the observer such that $\theta(n) = P(n)\phi(n)$. where $P(n)$ is a constant matrix of dimension $2n \times 2n$. The error can be defined as

$$e(n) = P(n)\phi(n) - \theta(n).$$

Hence

$$\begin{aligned} e(n+1) &= P(n+1)\phi(n+1) - \theta(n+1) \\ &= [P(n+1)B(n)-N(n)-L(n)D(n)]v(n) + [P(n+1)E(n)-M(n)- \\ &\quad L(n)H(n)]f(n) + [P(n+1)A(n)-L(n)C(n)]\phi(n)+G(n)\theta(n). \end{aligned}$$

We can find the homogeneous state equation for the error if the following equations are satisfied:

$$P(n+1)B(n) = N(n)+L(n)D(n)$$

$$P(n+1)E(n) = M(n)+L(n)H(n)$$

$$G(n)P(n) = P(n+1)A(n) - L(n)C(n).$$

In this case,

$$e(n+1) = G(n)e(n).$$

If we want the matrix $[P(n+1)A(n) - G(n)P(n)]$ to present quarter plane causality, then $P(n)$ should be subdiagonal. That is $P(n) = \mathrm{diag}[P_1 ; P_4]$. The uniqueness of $P(n)$ and existence of $P^{-1}(n)$ are discussed in appendix B.

Note that we must set the initial condition vector of the observer so that the result $\theta(n) = P(n)\phi(n)$ is obtained. That is $\theta(0) = P(0)\phi(0)$. It is obvious that it is not always possible to set these initial conditions correctly. Thus the norm of matrix $G(n)$ must be less than one. If the norm of $G(n)$ is very small, then any transient response due to an error will die out quickly. We must be able to invert matrix $P(n)$ to obtain $\phi(n)$. It is desirable to avoid this inversion , and thus we investigate the possibilities of setting $P(n) = I$, a $2n \times 2n$ identity matrix. In this case, we have the following set of equations:

$$B(n) = N(n) + L(n)D(n)$$

$$E(n) = M(n) + L(n)H(n)$$

$$G(n) = A(n) - L(n)C(n)$$

and the observer is described by

$$\theta(n+1) = [A(n) - L(n)C(n)]\theta(n) + B(n)v(n) + E(n)f(n) + L(n)C(n)\phi(n).$$

A possible disadvantage to this method is that all the state variables are produced by the observer, whereas some are already available and some are not needed. It is worthwhile to mention that method-1 reproduces only desired state variables. Now we can state the following theorem.

<u>Theorem (8)</u>: the state $\theta(n)$ is an estimate of $\phi(n)$ in the sense that $[\phi(n) - \theta(n)]$ goes to zero as n goes to ∞, for any $\phi(0)$, $\theta(0)$, and $v(n)$ if

1) $G(n) = A(n) - L(n)C(n)$
2) $M(n) = E(n)-L(n)H(n)$
3) $N(n) = B(n)-L(n)D(n)$
4) $\prod_{n=1}^{\infty} ||G(n)||$ goes to zero.

<u>Proof</u>: Consider error vector $e(n) = \phi(n) - \theta(n)$. If 1), 2), and 3) are satisfied we find that $e(n+1) = G(n)e(n) =[A(n) - L(n)C(n)]e(n)$. Thus by theorem 4.1, $e(n)$ is going to go to zero as n goes to ∞.∎

Since the matrices $G(n)$, $M(n)$, and $N(n)$ have the banded structure we can state the following equivalent theorem.

<u>Theorem (9)</u>: The state $\theta(n)$ is an estimate of $\phi(n)$ if

1) $G(i,j) = [A-LC](i,j)$
2) $M(i,j) = [E-LH](i,j)$
3) $N(i,j) = [B-LD](i,j)$
4) $\prod_{n=1}^{\infty} \bar{\sigma}_Q(n)$ goes to zero.

<u>Proof</u>: Use of theorem 8 and equation 10 gives the stated results.∎

<u>Algorithm</u>

(I) Find $L(i,j)$ such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_Q(n) \text{ goes to zero.}$$

(II) From 2) of theorem (9) find matrix $M(n)$

(III) From 3) of theorem (9) find matrix $N(n)$.

It should be noted that the main objective is to find $L(i,j)$ such that

$$\prod_{n=1}^{\infty} \bar{\sigma}_Q(n) \text{ goes to zero.}$$

Remark. It should be noted that condition 4 of theorems 8 and 9 can be substituted by $||G(n)|| < 1 - \varepsilon$, and $\sigma_Q(n) < 1-\varepsilon$ respectively, for all integer n and $\varepsilon > 0$.

Observation: It can easily be seen that the matrices $K(n)$, $R(n)$, $M(n)$, and $\{L(n), N(n)\}$ have the same structure of the matrices $C(n)$, $D(n)$, $E(n)$, and $B(n)$ of W-A model respectively. It is also clear that the matrices $[A(n)+B(n)K(n)]$, $[A(n)-L(n)C(n)]$, and $[A(n)+B(n)R(n)C(n)]$ have identical structures. Thus $[A(n)+B(n)R(n)C(n)]$ is the most general form. By choosing $R(n)C(n) = K(n)$ and $B(n)R(n) = -L(n)$ the other matrices are obtained.

6.4 RELATION OF STATE FEEDBACK AND STATE OBSERVER

Consider the Wave model equation

$$\phi(n+1) = A(n)\phi(n) + B(n)v(n) + E(n)f(n)$$

$$\mu(n) = C(n)\phi(n) + D(n)v(n) + H(n)f(n) \tag{16}$$

and state estimator

$$\theta(n+1) = G(n)\theta(n) + N(n)v(n) + M(n)f(n) + L(n)\mu(n) \tag{17}$$

define the state feedback

$$v(n) = u(n) - K(n)\theta(n) \tag{18}$$

substituting for $v(n)$ in equation 16 and 17 yields

$$\phi(n+1) = A(n)\phi(n) + B(n)u(n) - B(n)K(n)\theta(n) + E(n)f(n)$$

$$\theta(n+1) = G(n)\theta(n)+N(n)u(n)-N(n)K(n)\theta(n)+M(n)f(n)+L(n)C(n)\phi(n)$$

this can be written in the following matrix form

$$\begin{bmatrix} \phi(n+1) \\ \theta(n+1) \end{bmatrix} = \begin{bmatrix} A(n) & -B(n)K(n) \\ L(n)C(n) & G(n)-N(n)K(n) \end{bmatrix} \begin{bmatrix} \phi(n) \\ \theta(n) \end{bmatrix} + \begin{bmatrix} B(n) \\ N(n) \end{bmatrix} u(n)$$

$$+ \begin{bmatrix} E(n) \\ M(n) \end{bmatrix} f(n) \qquad (19)$$

by considering

$$G(n) = [A(n) - L(n)C(n)], \quad B(n) = N(n)$$

and some manipulation the following state matrix will be obtained

$$\mathcal{F}(n) = \begin{bmatrix} A(n)-B(n)K(n) & -B(n)K(n) \\ 0 & A(n)-L(n)C(n) \end{bmatrix}.$$

Clearly, The observer error is completely decoupled and will go to zero, provided that the observer is stable. However, even if the state feedback matrix, $A(n) - B(n)K(n)$, is stable it is not in general possible to guarantee that $\phi(n)$ will tend to zero. The effect of the coupling term, $B(n)K(n)$, must be considered.

If norm criteria are used it can be seen that the above matrix, $\mathcal{F}(n)$, is norm invariant. Thus it is sufficient to consider the norm of

$$\mathcal{F} = \begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix}.$$

We calculate

$$\mathcal{f}^T\mathcal{f} = \begin{bmatrix} (A-BK)^T(A-BK) & (A-BK)^T(-BK) \\ (-BK)^T(A-BK) & (-BK)^T(-BK)+(A-LC)^T(A-LC) \end{bmatrix} .$$

It should be noted that the $\mathcal{f}^T\mathcal{f}$ is full although the matrix $\mathcal{f}$ is block triangular. However if $|\sigma_{max}(\mathcal{f}^T\mathcal{f}| < 1$ then

1) $|\sigma_{max}[(A-BK)^T(A-BK)]| < 1$

and

2) $|\sigma_{max}[(A-LC)^T(A-LC)+(-BK)^T(-BK)]| < 1$

condition 2) also implies

$$|\sigma_{max}[(A-LC)^T(A-LC)]| < 1$$

this is because the matrix $[(-BK)^T(-BK)]$ is non negative. The above results can be stated in the following form. If $||\mathcal{f}|| < 1$ then $||(A-BK)|| < 1$ and $||(A-LC)|| < 1$. Therefore if the system of equation 14 satisfies the one step stability test, then the state feedback system and the state estimator system are stable.

The general solution to the stability analysis of the state feedback with observer is a problem for future research. If stability of both observer and state feedback do not assure stability of the overall system this will mark another fundamental distinction between 1-D and m-D systems.

CHAPTER 7

CONCLUSION

A new approach to the analysis of stability for 2-D digital recursive filters has been presented. This approach, using the Wave model, enables one to use 1-D techniques for 2-D systems. Although necessary and sufficient stability conditions are established for special cases, the general case remains elusive and only sufficient conditions are available.

Our analysis ,using Lyapunov method on the Wave model, highlights the basic difficulty in the stability studies for m-D systems. In utilizing the 1-D nature of the Wave model, the Lyapunov equations are time variant, even for constant matrices in the G-R model. This time variant characteristic invalidates the standard necessary and sufficient conditions available for stationary, 1-D systems. However, as our study shows, it is relatively straight forward to generate necessary or sufficient conditions. Some of the conditions, in particular those of lemmas 4.11 and 4.12 and theorem 4.18 are unique to our approach in terms of Lyapunov functions.

In chapter 5 we consider the use of feedback to stabilize 2-D systems. In particular the use of the stability criteria established in chapter 4 is considered. Since sufficient conditions are often appropriate to the stabilization question, the criteria of chapter 4 are judged to be quite effective.

In chapter 6 the classic observer problem has been considered. Here one must specify a model (observer) system and couple it to the original system. The coupling must result in asymptotic slaving of the states in the two systems. Our interest has been in using the criteria of chapter 5 to insure stability of the coupled systems.

The studies of state feedback and state observer have shown the power of the Wave model in transporting 1-D results to the m-D setting.

APPENDIX A

In this appendix we explore the structure of the matrices $A(n+1)A(n)$, $\Phi(n;2)$, $A(n+2)A(n+1)A(n)$, and $\Phi(n;3)$ defined in the main body of chapter 4.

To identify the form of matrix $A(n+1)A(n)$ it suffices to examine the $n = 2$ case, namely

$$A(2)A(1) = \begin{bmatrix} A_4 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & A_3 & A_4 & 0 \\ 0 & A_1 & A_2 & 0 \\ 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix} \begin{bmatrix} A_4 & 0 \\ A_2 & 0 \\ 0 & A_3 \\ 0 & A_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_4{}^2 & 0 \\ A_2A_4 & 0 \\ A_3A_2 & A_4A_3 \\ A_1A_2 & A_2A_3 \\ 0 & A_3A_1 \\ 0 & A_1{}^2 \end{bmatrix}$$

It should be noted that the matrix $[A(n+1)A(n)]$ can be easily constructed from matrix $[A(2)A(1)]$. Actually they have the following form:

$$A(3)A(2) = \left[\begin{array}{c|c} A(2)A(1) & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & A(2)A(1) \end{array}\right]$$

and

$$A(4)A(3) = \left[\begin{array}{c|c|c} A(2)A(1) & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & A(2)A(1) & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & A(2)A(1) \end{array}\right].$$

The above cases emphasize the general pattern. To formalize our observation let $Z(0)$ be any matrix with a two block partition structure

$$Z(0) = \mathrm{col}[Z_1 : Z_2].$$

We define

$$\mathrm{band}_2\ [Z(0)] = \begin{bmatrix} Z_1 & 0 \\ Z_2 & Z_1 \\ 0 & Z_2 \end{bmatrix}$$

$$\mathrm{band}_n\ [Z(0)] = \begin{bmatrix} Z_1 & 0 & & 0 \\ Z_2 & Z_1 & & \\ 0 & Z_2 & & \vdots \\ \vdots & 0 & \cdot & \\ & & & Z_1 \\ 0 & 0 & & Z_2 \end{bmatrix}.$$

With this notation, the general form of the transition matrices can be easily established as follows:

$A(3)A(2) = \mathrm{band}_2\ [A(2)A(1)]$

$A(4)A(3) = \mathrm{band}_3\ [A(2)A(1)]$

:

$A(i+1)A(i) = \text{band}_i\ [A(2)A(1)]$.

It is straightforward to verify that $\Phi(2;2)$ will have the following structure:

$$\Phi(2;2) = \begin{bmatrix} G & H \\ H^T & F \end{bmatrix}$$

where

$$G=(A_4{}^2)^T A_4{}^2+(A_2A_4)^T(A_2A_4)+(A_3A_2)^T(A_3A_2)+(A_1A_2)^T(A_1A_2)$$

$$H=(A_3A_2)^T(A_4A_3)+(A_1A_2)^T(A_2A_3)$$

$$F=(A_1{}^2)^T A_1{}^2+(A_3A_1)^T(A_3A_1)+(A_2A_3)^T(A_2A_3)+(A_4A_3)^T(A_4A_3)$$

We note that at step n = 4 all the elements of the matrix $\Phi(n;2)$ are known therefore the matrix $\Phi(n;2)$ can be constructed. This is because as n increases the matrix $A(n+1)A(n)$ will obtain more zero elements and thus multiplying this matrix by its transpose will result in a banded structure matrix $\Phi(n;2)$.

The reader may also readily verify that the matrix $A(3)A(2)A(1)$ has the form

$$A(3)A(2)A(1) = \begin{bmatrix} A_4{}^3 & 0 \\ A_2A_4{}^2 & 0 \\ A_3A_2A_4+A_4A_3A_2 & A_4{}^2A_3 \\ A_1A_2A_4+A_2A_3A_2 & A_2A_4A_3 \\ A_3A_1A_2 & A_1A_2A_3+A_4A_3A_1 \\ A_1{}^2A_2 & A_1A_2A_3+A_2A_3A_1 \\ 0 & A_3A_1{}^2 \\ 0 & A_1{}^3 \end{bmatrix}$$

It can easily be seen that the matrix $A(n+2)A(n+1)A(n)$ is of the form

$$A(i+2)A(i+1)A(i) = \text{band}_i\ [A(3)A(2)A(1)].$$

Therefore having the elements of the matrix A(3)A(2)A(1) is sufficient for constructing the matrix A(n+2)A(n+1)A(n). The reason is that no new element appears as n increases.

Remark 1. Each column of the matrix A(n+2)A(n+1)A(n) has a maximum of six non zero elements whereas the matrix A(n+1)A(n) had four non zero elements per column. By some manipulation it easily can be seen that each column of the matrix A(n+k)A(n+k-1).....A(n) will have a maximum of 2k non zero elements.

Knowing the structure of the matrix A(n+2)A(n+1)A(n) the matrix $\Phi(n;3)$ can be easily constructed. $\Phi(n;3)$ has the following form

$$\Phi(n;3) = [A(n+2)A(n+1)A(n)]^T[A(n+2)A(n+1)A(n)]$$

$$= \begin{bmatrix} X & Y & Z & W & 0 & 0 & 0 & \cdots & 0 \\ Y^T & X & Y & Z & W & 0 & 0 & \cdots & 0 \\ Z^T & Y^T & X & Y & Z & W & 0 & \cdots & 0 \\ W^T & Z^T & Y^T & X & Y & Z & W & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & W \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & Z \\ & & & & & & & & Y \\ 0 & 0 & 0 & 0 & 0 & \cdots & W^T & Z^T\ \ Y^T & X \end{bmatrix}$$

where

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_4 \end{bmatrix}$$

$$Z = \begin{bmatrix} Z_1 & 0 \\ Z_3 & Z_4 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & 0 \\ W_3 & 0 \end{bmatrix}.$$

It can easily be shown that the matrix $\Phi(n;3)$ is a symmetric Toeplitz matrix with four diagonals. The elements of the diagonal are themselves matrices and can be easily calculated.

<u>Remark 2.</u> The matrices $\Phi(n;1)$, $\Phi(n;2)$, ...$\Phi(n;i)$... have dimension 2n x 2n. Therefore only 2n eigenvalues are calculated for any i-step test. This is because the matrix $\Phi(n;i+1)$ has the form $A^T(n)E_i(n)A(n)$ where, $E_i(n) = [A(n+i)....A(n-1)]^T[A(n+i)....A(n-1)]$.

<u>Remark 3.</u> Note that having the elements of the matrices A(1), A(2)A(1), A(3)A(2)A(1),......., A(i)A(i-1)A(i-2)....A(1) is enough for constructing the following matrices, A(n), A(n+1)A(n), A(n+2)A(n+1)A(n),......., A(n+i-1)A(n+i-2)A(i-2)....A(n) respectively. Therefore the matrices $\Phi(n;1)$, $\Phi(n;2)$, $\Phi(n;3)$,......., $\Phi(n;i)$ can be easily determined. Moreover matrices $\Phi(n;1)$, $\Phi(n;2)$, $\Phi(n;3)$,......., $\Phi(n;i)$ can be structed directly after calculating the matrices A(2), A(4)A(3), A(6)A(5)A(4),

........, $A(2i)A(2i-1).....A(i+1)$. It is also interesting to note that the matrices $\Phi(n;i+1)$ are symmetric Toeplitz matrices with $(i+1)$ diagonals. The elements of these diagonals are also matrices.

Remark 4. It should be noted that if the maximum eigenvalue of the matrices $\Phi(2;1)$, $\Phi(3;2)$, $\Phi(4;3)$,......., $\Phi(i+2;i+1)$ for all $i \le n$ is greater than unity the i-step test fails. This is because the above matrices are submatrices of the matrices $\Phi(n;1)$, $\Phi(n;2)$, $\Phi(n;3)$,......., $\Phi(n;i+1)$ respectively. In addition by theorem 4.5 the maximum eigenvalue of $\Phi(n;i+1)$ is greater than or equal to maximum eigenvalues of $\Phi(i+2;i+1)$. Therefore $\sigma_{max}[\Phi(i+2;i+1)] \ge 1$ implies $\sigma_{max}[\Phi(n;i+1)] \ge 1$. That is the system stability can not be determined.

Finally we compare each i-step test and emphasize the physical meaning behind it. For the 1-step test we consider only one sequence of state $\phi(i)$. In this case the condition

$$||\Phi(n;1)|| < 1$$

implies that

$$||\Phi(1)|| > ||\Phi(2)|| > ||\Phi(3)|| > ....$$

that is as n increases the norm of the state decreases. In 2-step test the following two set of sequences of state are considered. In this case $||\Phi_1(n)|| < 1$ implies that

$$||\Phi(1)|| > ||\Phi(3)|| > ||\Phi(5)|| > ....$$

$$||\Phi(2)|| > ||\Phi(4)|| > ||\Phi(6)|| > .... \quad .$$

Clearly for the i-step test we have i sets of sequences and the condition $||\Phi_{i-1}(n)|| < 1$ implies that

$$||\Phi(1)|| > ||\Phi(i+1)|| > ||\Phi(2i+1)|| > \ldots\ldots\ldots\ldots$$

$$||\Phi(2)|| > ||\Phi(i+2)|| > ||\Phi(2i+2)|| > \ldots\ldots\ldots\ldots$$

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$$||\Phi(k)|| > ||\Phi(i+k)|| > ||\Phi(2i+k)|| > \ldots\ldots\ldots\ldots .$$

Note that if in each test at least one of the inequality is not satisfied the rest of the inequalities are not going to be satisfied. Thus it is not necessary to check them.

APPENDIX B

SOLUTION OF THE MATRIX EQUATION

[P(n+1)A(n)-G(n)P(n)=L(n)C(n)]

In this section we discuss the existence and uniqueness of the matrix P(n). We also state some conditions for existence of $P^{-1}(n)$.

Due to the banded structure of matrices A(n), G(n) and block diagonality of matrix P(n) the above equation is equivalent to the following:

$$P_1A_4 - G_4P_1 = L_1C_2 \tag{1}$$

$$P_1A_3 - G_3P_4 = L_1C_1 \tag{2}$$

$$P_4A_2 - G_2P_1 = L_2C_2 \tag{3}$$

$$P_4A_1 - G_1P_4 = L_2C_1. \tag{4}$$

The four equations can be put in the following compact form

$$\bar{P}A - G\bar{P} = \bar{L}C \; ; \tag{5}$$

where

$$\bar{P} = \begin{bmatrix} P_4 & 0 \\ 0 & P_1 \end{bmatrix} ,$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} ,$$

$$G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix},$$

$$\bar{L} = [L_2 \quad L_1],$$

$$C = [C_1 \quad C_2].$$

From [86], equation 1 has a unique solution P_1 if A_4 and G_4 have no common eigenvalues. Similarly equation 4 has a unique solution P_4 if A_1 and G_1 have no common eigenvalues.

Assuming A_4 and G_4 (similarly A_1, G_1) do not have any common eigenvalues. Then the necessary conditions for existence of $P^{-1}{}_1$ (similarly $P^{-1}{}_4$) are the following

i) $\{A_4 \ C_2\}$ is observable ($\{A_1 \ C_1\}$ is observable).

ii) $\{G_4 \ L_1\}$ is controllable ($\{G_1 \ L_2\}$ is controllable).

Therefore we should do the following

Algorithm

1) check if i) is satisfied. If not P^{-1} does not exist
2) choose G_1 and G_4 such that G_1 and A_1, G_4 and A_4 have no common eigenvalues
3) choose L such that ii) is satisfied
4) solve equation 1 and 4 for P_1 and P_4
5) solve equation 2 and 3 for G_2 and G_3
6) check if $||G|| < 1$. If not, choose another G_1 and G_4 and try step 2-6 again.

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